# Unit - III

## Fourier Series and Sequence & Series

**Periodic functions:-** A function $f(x)$ is said to be periodic with period T if

$$f(x+T) = f(x), \text{ for all } x.$$

Where T is the smallest +ve number.

Thus if T is period of $f(x)$, then

$$f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT) = \cdots$$

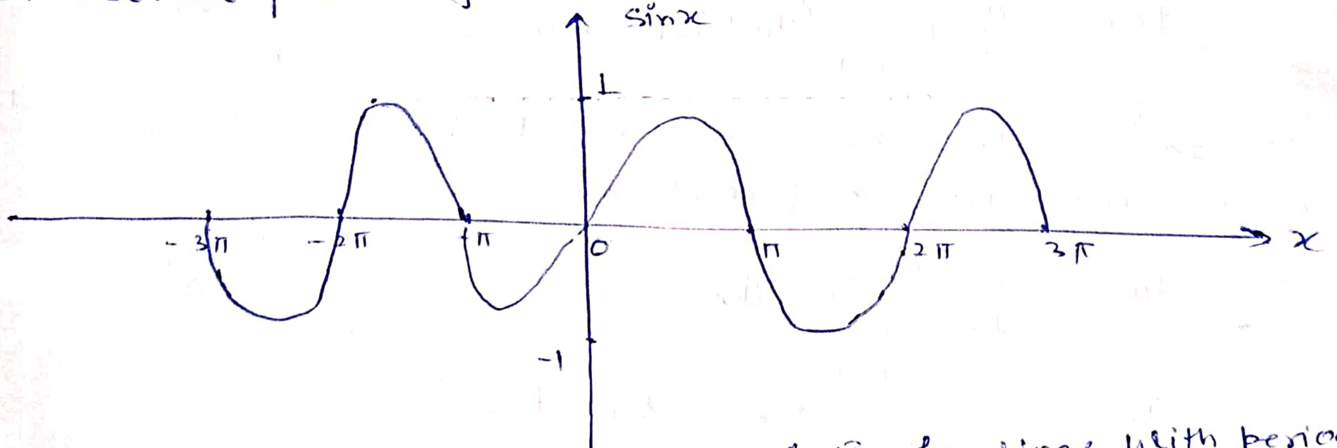
Also $f(x) = f(x-T) = f(x-2T) = \cdots = f(x-nT) = \cdots$

$\therefore$ $f(x) = f(x \pm nT)$ ; where n is +ve integer.

Thus $f(x)$ repeates itself after period of T.

For example (1) $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots$

So $\sin x$ is periodic with period $2\pi$. This is called Sinusoidal peridic function.




Also $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic functions with period $2\pi$

(2) Also Again $\tan(x+\pi) = \dfrac{\sin(x+\pi)}{\cos(x+\pi)} = \dfrac{-\sin x}{-\cos x} = \tan x$

$\cot(x+\pi) = \dfrac{\cos(\pi+x)}{\sin(\pi+x)} = \dfrac{-\cos x}{-\sin x} = \cot x$

$\therefore$ $\cot x$ & $\tan x$ are periodic function with period $\pi$.

Remark (1) $\sin nx$ and $\cos nx$ are periodic function with period $\frac{2\pi}{n}$.

(2) The sum of a finite no. of periodic function is periodic. If $T_1$ & $T_2$ are periods of $f(x)$ & $g(x)$ then period of $af(x) + bg(x)$ is L.C.M. of $T_1$ & $T_2$.

example $\cos x, \cos 2x, \cos 3x$ are periodic function with period $\frac{2\pi}{1}, \frac{2\pi}{2}, \frac{2\pi}{3}$ respectively.

$f(x) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ is also periodic function with period $2\pi$ which is l.c.m. of $2\pi, \frac{2\pi}{2}, \frac{2\pi}{3}$

$$= \pi\left[2, 1, \frac{2}{3}\right] = \pi\cdot\left[\frac{L.C.M \text{ of Numerator}}{H.C.M \text{ of denominator}}\right] = \pi\left[\frac{L.C.M(1,2,2)}{H.C.M(1,3)}\right] = 2\pi.$$

## Generalized rule of Integral by parts

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$ where $u$ & $v$ are functions of $x$. and

$u' = \frac{du}{dx}$ and $v_1 = \int v\,dx$

$u'' = \frac{du'}{dx}$ $\quad v_2 = \int v_1\,dx$

$u''' = \frac{du''}{dx}$ $\quad v_3 = \int v_2\,dx$

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**Fourier series :-** Fourier series for the function $f(x)$ in the interval $c < x < c+2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)} \quad \text{—}(1)$$

where $a_0$, $a_n$ & $b_n$ are called Fourier coefficients, these values are given by Euler's as

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

are known as Euler's formulae.

**To find $a_0$** Int. both side of (1), w.r.t. $x$ between the limits $c$ to $c+2\pi$

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} 1\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n\cos nx\right)dx$$

$$\therefore \int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}(\cancel{c}+2\pi-\cancel{c}) + 0 + 0.$$

$$\because \int_c^{c+2\pi} \sin nx\,dx = \int_c^{c+2\pi} \cos nx\,dx = 0 \quad (n \neq 0)$$

$$\therefore \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

<u>To find $a_n$</u>:- multiplying both sides by $\cos nx$ and int. w.r.to $x$ between the limits $c$ to $c+2\pi$

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\cos nx\,dx$$

$$= 0 + a_n\pi + 0$$

$$\therefore \int_c^{c+2\pi}\sin nx\cos nx\,dx = 0$$

$$\int_c^{c+2\pi}\sin^2 nx\,dx \text{ or } \cos^2 nx\,dx = \pi.$$

$$\therefore \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx}$$

<u>To find $b_n$</u>:- multiplying both sides by $\sin nx$ and int. w.r.to $x$ between the limit $c$ to $c+2\pi$, we get

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= \frac{a_0}{2}\times 0 + a_n\times 0 + b_n\times\pi$$

$$\therefore \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

The values of $a_0$, $a_n$ and $b_n$ are called Euler's formulae

<u>Remark</u> ① If $c=0$, then interval becomes $0<x<2\pi$ and Euler's <u>or case-II</u> formulae are $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

**Case-II** If $c = -\pi$ then interval becomes $-\pi < x < \pi$ and formulae are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

There are two cases aries.

**Subcase-I** When $f(x)$ is an odd function i.e. $f(-x) = -f(x)$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ ; $f(-x) = f(x)$ even fun.

$= 0$ ; $f(-x) = -f(x)$ odd fun.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \; ; \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

The above series is known as Fourier sine series or half range Fourier sine series.

**Subcase-II** When $f(x)$ is even function i.e. $f(-x) = f(x)$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0.$$

Hence Fourier series becomes

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

which is known as Fourier cosine series or Half Range Fourier cosine series. (5)

**Dirichlet's Condition:-** Any function $f(x)$ can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(\cos nx + b_n \sin nx)$ where $a_0$, $a_n$ & $b_n$ are constants, provided

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite discontinuities in any one period.

(iii) $f(x)$ has a finite number of maxima and minima.

**Que 1** Obtain the Fourier series to represent $f(x) = \frac{1}{4}(\pi - x)^2$ in the interval $0 \le x \le 2\pi$. Hence obtain the following relations

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \infty = \frac{\pi^2}{6}$.

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \infty = \frac{\pi^2}{12}$.

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \infty = \frac{\pi^2}{8}$.

(IV) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24}$.

**Sol:-** Let $f(x) = \frac{1}{4}(\pi - x)^2$ ; $0 \le x \le 2\pi$.

we have Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \text{——— (1)}$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{1}{4\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{\pi^2}{6}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\left\{(\pi - x)^2\right\}\left(\frac{\sin nx}{n}\right) - \left\{2(\pi - x)(-1)\right\}\left(\frac{-\cos nx}{n^2}\right) + \left\{2(-1)(-1)\right\}\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^2\sin nx}{n} + \frac{2(\pi-x)\cos nx}{n^2} - \frac{2}{n^3}\sin nx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left\{\frac{(-\pi)^2\sin 2n\pi}{n} + \frac{-2\pi\cos 2n\pi}{n^2} - \frac{2}{n^3}\sin 2n\pi\right\} - \left\{0 + \frac{2\pi}{n^2} - 0\right\}\right]$$

$$= \frac{1}{4\pi}\left[+\frac{2\pi}{n^2}(-1)^{2n} + \frac{2\pi}{n^2}\right] = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi-x)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - \{2(\pi-x)(-1)\}\left(\frac{-\sin nx}{n^2}\right) + \{2(-1)(-1)\}\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{(\pi-x)^2\cos nx}{n} + \frac{2(\pi-x)\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left\{-\frac{\pi^2\cos 2n\pi}{n} - \frac{2\sin nx}{n^2} + \frac{2\cos 2n\pi}{n^3}\right\} - \left\{-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right\}\right]$$

$$= \frac{1}{4\pi}\left[\cancel{-\frac{\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}}\right]$$

$$= 0.$$

By equation (1), we get

$$f(x) = \frac{1}{2}\left(\frac{\pi^2}{6}\right) + \sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} + 0\right)$$

$$\frac{1}{4}(\pi-x)^2 = f(x) = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \cdots \infty \quad \text{--- (2)}$$

(i) Putting $x=0$ in eq (2), we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

$$\Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \infty = \frac{\pi^2}{6}} \quad \text{--- (3)}$$

(ii) Putting $x=\pi$ in eq (2), we get

$$0 = \frac{\pi^2}{12} + \left[\frac{(-1)}{1^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \frac{(-1)^4}{4^2} + \frac{(-1)^5}{5^2} + \frac{(-1)^6}{6^2} \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots \infty = \frac{\pi^2}{12} \quad \text{--- (4)}$$

(iii) Adding (3) & (4), we get

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ \cdots + \cdots\right) = \frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ \cdots \infty = \frac{\pi^2}{8}}$$

(iv) Subtracting (4) from (3), we get

$$\boxed{\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+ \cdots \infty = \frac{\pi^2}{24}}$$

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Que 2| Expand $f(x) = x\sin x$, $0<x<2\pi$ as a Fourier Series

Sol:- Let $f(x) = x\sin x$ ; $0<x<2\pi$

Fourier series is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) \quad \text{——— (1)}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[(-2\pi\cos 2\pi + \sin 2\pi) - (0)\Big]$$

$$= -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx \quad \text{——— (2)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2}+\frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$(\because n+1 \neq 0)$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1}+\frac{\cos 2(n-1)\pi}{n-1}\right\} - 0\right]$$

$$= -\frac{1}{n+1}+\frac{1}{n-1} = \frac{2}{n^2-1}, \quad (n^2-1 \neq 0 \Rightarrow n^2 \neq 1)$$

When $n=1$, then we have by eq (2),

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$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\, 2\sin x \cos x \, dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$
$$= \frac{1}{2\pi}\left[ x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{2^2}\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[\left(-\frac{2\pi\cos 4\pi}{2} + \frac{\sin 4\pi}{2^2}\right) - (0)\right] = -\tfrac{1}{2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx \quad \text{———} \quad (3)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin nx \sin x)\, dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi} x\left\{\cos(n-1)x - \cos(n+1)x\right\} dx$$
$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - (1)\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[2\pi\left\{\frac{\sin 2(n-1)\pi}{n-1} - \frac{\sin(n+1)2\pi}{n+1}\right\} - \left\{-\frac{\cos 2(n-1)\pi}{(n-1)^2} + \frac{\cos(n+1)2\pi}{(n+1)^2}\right\} - \left\{0 + \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right\}\right]$$
$$= \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n-1)^2}} + \cancel{\frac{1}{(n+1)^2}}\right] = 0 \;; \quad n-1 \neq 0.$$

When $n=1$, then by eq (3), we have

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\left\{1 - \cos 2x\right\} dx$$
$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{2^2}\right)\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[2\pi\left(2\pi - \frac{\sin 4\pi}{2}\right) - \left(\frac{4\pi^2}{2} + \frac{\cos 4\pi}{2^2}\right) - \left\{0 - \frac{\cos 0}{2^2}\right\}\right]$$
$$= \frac{1}{2\pi}\left[4\pi^2 - 2\pi^2 - \cancel{\tfrac{1}{2^2}} + \cancel{\tfrac{1}{2^2}}\right]$$
$$= \frac{1}{2\pi}\left[2\pi^2\right] = \pi.$$

Que 3) Find the Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = +\pi$. Hence show that (9)

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots = \frac{\pi^2}{12}.$$

Sol:- $f(x) = x - x^2$, $\quad : \quad -\pi \leq x \leq \pi$.

Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——— (1)}$$

Where
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2)\, dx$$

$$= \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} x\, dx - \int_{-\pi}^{\pi} x^2\, dx \right]$$

$$= \frac{1}{\pi} \left[ 0 - 2 \int_{0}^{\pi} x^2\, dx \right]$$

$$= \frac{1}{\pi} \left[ -2 \times \left( \frac{x^3}{3} \right)_0^{\pi} \right] = -\frac{2}{3}\pi^2.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} \underbrace{x \cos nx}_{\text{Odd}}\, dx - \int_{-\pi}^{\pi} \underbrace{x^2 \cos nx}_{\text{Even}}\, dx \right]$$

$$= \frac{1}{\pi} \left[ 0 - 2 \int_0^{\pi} x^2 \cos nx\, dx \right]$$

$$= -\frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\, dx$$

$$= -\frac{2}{\pi} \left[ x^2 \left( \frac{\sin nx}{n} \right) - (2x) \left( -\frac{\cos nx}{n^2} \right) + (2) \left( -\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= -\frac{2}{\pi} \left[ \frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi}$$

$$= -\frac{2}{\pi} \left[ \left( 0 + \frac{2\pi (-1)^n}{n^2} - 0 \right) - (0) \right]$$

$$= -\frac{4}{n^2} (-1)^n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \sin nx\, dx - \int_{-\pi}^{\pi} x^2 \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[2\int_0^{\pi} x \sin nx\, dx - 0\right]$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - (0)\right]$$

$$= -\frac{2}{n}(-1)^n$$

Put these values in eq (1), we get

$$f(x) = \frac{1}{2}\left(-\frac{2}{3}\pi^2\right) - \frac{4}{1}\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n}$$

$$x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n}\cos nx}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n}.$$

Put $x = 0$

$$0 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n \cdot 1}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin 0}{n}.$$

$$0 = -\frac{\pi^2}{3} - \frac{4}{1}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2} - 0$$

$$\therefore \frac{\pi^2}{3} = -4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

Que 3) Find the Fourier series for the function $f(x) = x + x^2$, $-\pi < x < \pi$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \infty = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \infty = \frac{\pi^2}{12}.$

**Que 4)** Express $f(x) = |x|$ ; $-\pi < x < \pi$, as Fourier series. Hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

**Que 5)** Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. Deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi - 2}{4}.$$

**Que 6)** Find the Fourier series to represent $e^x$ in the interval $-\pi < x < \pi$

**Que 7)** Find the fourier series representation of the following Functions (i) $|\cos x|$ ; $-\pi < x < \pi$

(ii) $|\sin x|$ ; $-\pi < x < \pi$

## Fourier Series for discontinuous functions

**Que 8)** Find the fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } 0 \le x \le 2\pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$

Sol:- Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ —(1)

where
$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\pi\times 0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right) + \left(0 - \frac{(-1)^{2n}}{n^2}\right) - \left(\pi\times 0 - \frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] = \begin{cases} -\frac{4}{n^2\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right\}_0^{\pi} + \left\{-\frac{(2\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi(-1)^n}{n} + 0\right) - (0) + (0 - 0) - \left(\frac{-\pi\cos n\pi}{n} - 0\right)\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\pi(-1)^n}{n}} + \cancel{\frac{\pi}{n}(-1)^n}\right] = 0.$$

Put these values in eq (1), we get

$$\boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)}$$

Ques 9) Obtain the Fourier series for the function

$$f(x)=\begin{cases} x & -\pi<x<0 \\ -x & 0<x<\pi \end{cases}$$

and hence show that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\infty=\frac{\pi^2}{8}.$$

Que 10) Find the Fourier series expansion for the function

$$f(x)=\begin{cases} -1 & : \quad -\pi<x<-\pi/2 \\ 0 & : \quad -\pi/2<x<\pi/2 \\ 1 & : \quad \pi/2<x<\pi \end{cases}$$

Hence deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\cdots$

## Change of Variable

- Fourier series $f(x)$ in the arbitrary interval $c<x<c+2\pi$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right\}$$

where
$$a_0=\frac{1}{l}\int_c^{c+2l}f(x)\,dx$$

$$a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

- Case-I when $c=0$ then interval becomes $0<x<2l$ and above formula becomes

$$a_0=\frac{1}{l}\int_0^{2l}f(x)\,dx$$

$$a_n=\frac{1}{l}\int_0^{2l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n=\frac{1}{l}\int_0^{2l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Case-II when $c=-l$ then interval becomes $-l<x<l$ and above results becomes –

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad \& \quad b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Subcase-I when $f(x)$ is even function i.e $f(-x) = f(x)$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = 0.$$

and fourier series becomes

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)}$$

which is known as Fourier cosine series or Half Range Fourier cosine series.

Sub case - II. when $f(x)$ is odd function i.e $f(-x) = -f(x)$

$$\therefore \quad a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

and fourier series becomes

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)}$$

which is known as Fourier sine series or Half range Fourier sine series.

Que 1| Obtain the Fourier series expansion of

$$f(x) = \left(\frac{\pi - x}{2}\right) \quad \text{for} \quad 0 < x < 2.$$

sol:- let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right\}$ — (1)

Here the given range is $0 < x < 2$, which is of the form $0 < x < 2l$.

$$\therefore \; 2l = 2$$

$$\boxed{l = 1}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{1}\int_0^2 \left(\frac{\pi - x}{2}\right)dx = \left[\frac{1}{2}\left(\pi x - \frac{x^2}{2}\right)\right]_0^2$$

$$= \frac{1}{2}\left[2\pi - 2\right] = \pi - 1.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\int_0^{2} f(x)\cos(n\pi x)\,dx = \int_0^{2}\left(\frac{\pi-x}{2}\right)\cos(n\pi x)\,dx$$

$$= \frac{1}{2}\left[(\pi-x)\left\{+\frac{\sin n\pi x}{n\pi}\right\} - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= \frac{1}{2}\left[\frac{(\pi-x)\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_0^2$$

$$= \frac{1}{2}\left[\left\{\frac{(\pi-2)\sin 2n\pi}{n\pi} - \frac{\cos 2n\pi}{n^2\pi^2}\right\} - \left\{0 - \frac{\cos 0}{n^2\pi^2}\right\}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2}\right] = 0.$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\int_0^{2}\left(\frac{\pi-x}{2}\right).\sin n\pi x\,dx$$

$$= \frac{1}{2}\left[(\pi-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= \frac{1}{2}\left[-\frac{(\pi-x)\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^2$$

$$= \frac{1}{2}\left[\left\{-\frac{(\pi-2)\cos 2n\pi}{n\pi} - \frac{\sin 2n\pi}{n^2\pi^2}\right\} - \left\{-\frac{\pi}{n\pi} - 0\right\}\right]$$

$$= \frac{1}{2}\left[-\frac{(\pi-2)}{n\pi} + \frac{\pi}{n\pi}\right] = \frac{1}{2}\left[\frac{-\pi+2+\pi}{n\pi}\right] = \frac{1}{n\pi} \checkmark$$

From Eq (1), we get

$$\boxed{f(x) = \left(\frac{\pi-1}{2}\right) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}}$$ Ans

**Que 2)** Find the fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$.

**Que 3)** Obtain the Fourier series for $f(x) = \begin{cases} \pi x & ; 0 \le x \le 1 \\ \pi(2-x) & ; 1 \le x \le 2. \end{cases}$

**Que 4)** Find the Fourier series expansion of $f(x) = 1 + |x|$ defined in $-3 < x < 3$.

# Half Range Series

## Half Range Cosine series for $0<x<l$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)$$

where $a_0=\frac{2}{l}\int_0^l f(x)\,dx$

and $a_n=\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

For the interval $0<x<\pi$, we put $l=\pi$, then we get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

where $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$

$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos(n\pi)dx$

## Half-Range Fourier Sine series for $0<x<l$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right).$$

where $b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

For the interval $0<x<\pi$, we put $l=\pi$ in above we get

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

where $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$

---

**Que 1)** Expand $f(x)=x$ as a half Range
(i) sine series in $0<x<2$
(ii) cosine series in $0<x<2$.

**Sol:-** Here given range is of the form $0<x<l$. $\therefore l=2$.

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right). \quad \text{——} (1)$$

where $b_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$

$$= \frac{2}{2}\int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[ x\left\{\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right\} - (1)\left\{\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{2^2}}\right\}\right]_0^2$$

$$= \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \left[-\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\sin(n\pi) - 0\right]$$

$$= -\frac{4}{n\pi}(-1)^n$$

From (1) we have

$$\boxed{x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)}$$

(ii) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right). \quad \text{———— (2)}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{2}\int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2.$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[ x\left\{\frac{+\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right\} - (1)\left\{\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right\}\right]_0^2$$

$$= \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \left[\frac{4}{n\pi}\sin n\pi + \frac{4}{n^2\pi^2}\cos n\pi - 0 - \frac{4}{n^2\pi^2}\right]$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right].$$

Hence by eq (2) we get

$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos\left(\frac{n\pi x}{2}\right).$$ Ans

Que 2) Find the series of cosines of multiples of $x$ which (18). will represent $x \sin x$ in the interval $(0, \pi)$ and show that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi - 2}{4}$.

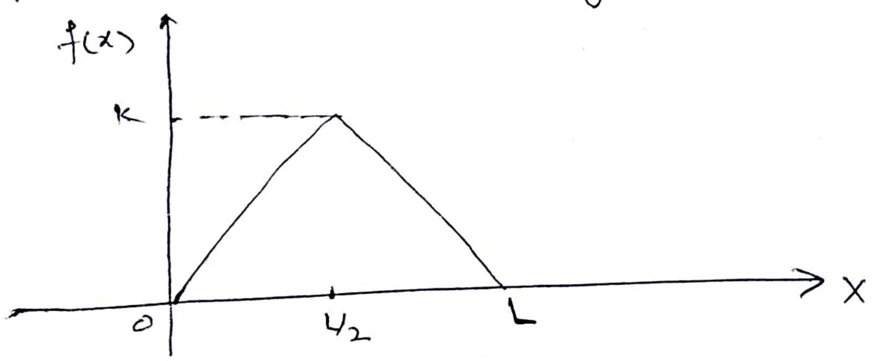
Que 3) Develop $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in half range cosine series in the range $0 < x < l$. Graph the corresponding periodic continuation of $f(x)$.

Que 4) Let
$$f(x) = \begin{cases} \omega x & \text{when } 0 \le x \le l/2 \\ \omega(l - x) & \text{when } l/2 \le x \le l. \end{cases}$$

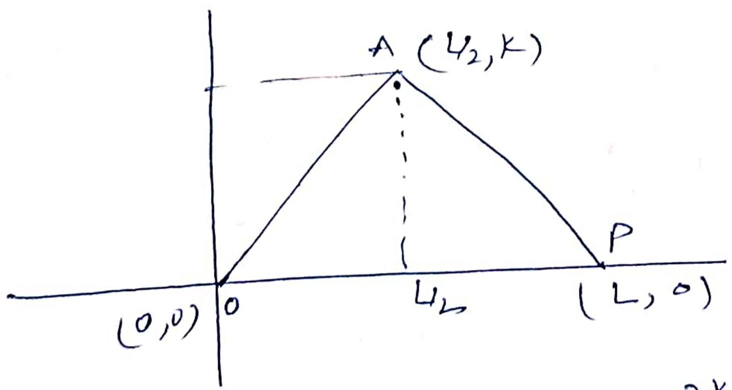
Show that
$$f(x) = \frac{4\omega l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{l}.$$

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Que 5) Find the half Range expansion of the function whose graph is given in following figure.




Sol:-




OA line $\quad y - 0 = \frac{K - 0}{\frac{L}{2} - 0}(x - 0) = \frac{2K}{L}(x)$

$$\therefore f(x) = \begin{cases} \frac{2K}{L} x & ; 0 < x < L/2 \\ \frac{2K}{L}(L - x) & ; \frac{L}{2} < x < L. \end{cases}$$

# Unit-3 (Sequence & series)

**Sequence:-** A sequence of real numbers (or real sequence) is a function $f: N \longrightarrow \mathbb{R}$, whose domain is the set of natural numbers and range a subset of real numbers.

If $f: N \rightarrow \mathbb{R}$ is a sequence then $\forall n \in N$, $f(n)$ is a real number.

It is customary to write $f(n)$ as $f_n$. A sequence may be written as $< f_1, f_2, \text{---} f_n \text{---}>$ or $< f_n >$ or $\{f_n\}$

The real numbers $f_1, f_2, \text{----} f_n \text{----}$ are called Ist, 2nd, ------ nth --- terms of the sequence.

The mth and nth terms $f_m$ & $f_n$ for $m \neq n$ are treated as distinct terms even if $f_m = f_n$. Thus the term of sequence occurring at different positions are treated as distinct terms even if they have the same value.

**Notation:-** If $f: N \rightarrow \mathbb{R}$ is a seq. of real numbers and it may be written as $< f(n) > = < f(1), f(2), \text{---} >$. We write $<f_n> = <a_n>, <b_n>, <u_n>, <v_n>$ etc.

$\forall n \in N$, $<a_n> = < a_1, a_2, \text{---} a_n \text{---}>$.

**Example** ① $<a_n> = <\frac{1}{n}> = <1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \text{------}>$.

② $< \frac{1}{2^{n-1}} > = < 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \text{------} >$.

③ $< (-1)^n > = < 1, -1, 1, -1, 1, -1 \text{---}>$.

④ $< 1+(-1)^n > = < 0, 2, 0, 2, 0, 2 \text{------}>$.

⑤ A seq. $<a_n>$ may be defined by recursion formula

$$a_{n+1} = \sqrt{2a_n}, \quad a_1 = 1.$$

The terms of seq. are $1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}} \text{---}$

⑥ $<a_n> = 2012, \ \forall n \in N.$

**Range of Sequence:-** The range of a sequence, consisting of all distinct elements of sequence and without regard to the

Position term. Thus range of a sequence may be finite (2)
or infinite set.

The range sets of above sequences are

(i) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \text{------} \}$

(ii) $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \text{------} \}$

(iii) $\{1, -1\}$

(iv) $\{0, 2\}$

(v) $\{1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}} \text{---} \}$

(vi) $\{2012\}$

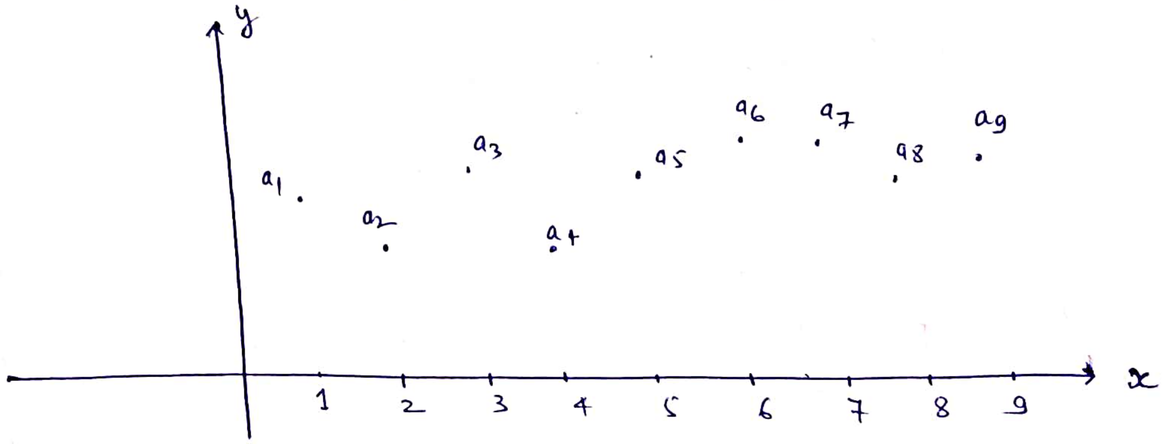
Constant Seq :- A seq $\langle a_n \rangle$ defined by $a_n = c \; \forall \; n \in N$, is called constant seq.

Exp $\{a_n\} = \{c, c, \text{--} c \text{..}\}$ is constant seq. with range $= \{c\}$.

Null seq :- A seq $\langle a_n \rangle$ defined by $a_n = 0 \; \forall \; n \in N$, is called Null seq

Note

A sequence is denoted by $\langle a_n \rangle = \{a_n\}$ whose ordinate $y = a_n$ at the abscissa $x = n$. Thus in a sequence for each $n \in N$, a number $a_n$ is assigned and is denoted as $\langle a_n \rangle$ or $\{a_n\}$ or $(a_n)$ or $\{a_1, a_2, \text{------} a_n \text{----}\}$




Infinite seq :- A seq. in which number of terms is infinite, is called infinite seq. and it is denoted by $\{a_n\}_{n=1}^{\infty}$. On the other hand finite seq. denoted by $\{a_n\}_{n=1}^{m}$ contains only a finite no. of terms ($m$ = finite).

**Bounded Seq:-** A seq $\langle a_n \rangle$ is said to be bounded if it's range set is bounded. Thus the seq $\langle a_n \rangle$ is bounded if there exists real numbers $m$ and $M$ such that

$$m < a_n < M \quad \forall \; n \in N.$$

Otherwise it is said to be unbounded.

**Monotonic seq:-** A seq $\langle a_n \rangle$ is said to be

(i) monotonically increasing if $a_{n+1} \geq a_n$ for every $n \in N$.
(ii) monotonically decreasing if $a_{n+1} \leq a_n$ for every $n \in N$.
(iii) Monotonic if it is either monotonically increasing or monotonically decreasing.

**Example** ① $\langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \text{----} \rangle$ bounded seq. because $0 < a_n = \frac{1}{n} < 1$ and Monotonically decreasing.

② $\{2^n\} = \{2, 2^2, 2^3, \text{---}\}$ unbounded seq. Since $2^n$ is larger and larger as $n$ comes larger and monotonically increasing seq.

**Limit of a sequence:-**

Consider a seq $\{a_n\} = \{3 + \frac{1}{n}\}$

Plotting the values

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$: | 1 | 2 | 4 | 5 | 10 | 50 | 100 | 1000 | 10000 | 100000 ---- |
| $a_n$: | 4 | 3.5 | 3.25 | 3.2 | 3.1 | 3.02 | 3.01 | 3.001 | 3.0001 | 3.00001 ---- |

As $n$ increases, $a_n = 3 + \frac{1}{n}$ becomes closer to 3.

Thus, the difference (or distance) between $3+\frac{1}{n}$ and 3 becomes smaller and smaller as $n$ becomes larger and larger i.e. we can ~~take~~ make $3+\frac{1}{3}$ and 3 as close as we please, by choosing an appropriately (sufficiently) large value for $n$, i.e. terms of seq cluster around this (limit) point. However note that $3+\frac{1}{n} \neq 3$ for any value of $n$.

**Limit:** — A number $l$ is said to be limit of a seq $\langle a_n \rangle$ and is denoted by as

$$\lim_{n\to\infty} a_n = \lim a_n = l$$

if for every $\epsilon > 0$, $\exists$ a +ve number $m$ s.t

$$|a_n - l| < \epsilon, \; \forall \; n \geq m.$$

**Remark:** — ① A seq may have a unique limit or may have more than one limit or may not have a limit

② If a seq. has a unique limit, we say that seq. is Convergent. Otherwise divergent.

**Convergence, divergence and oscillation of a seq.**

**Convergent:** — A seq $\langle a_n \rangle$ is said to be convergent if it has a finite limit i.e $\lim_{n\to\infty} a_n = l$ = finite unique limit value

**Divergent:** — If $\lim_{n\to\infty} a_n$ = infinite $= \pm\infty$.

**Oscillation:** If limit of $a_n$ is not unique (oscillates finitely) or $\pm\infty$ (oscillates infinitely).

**Exp** ① $\{\frac{1}{n^2}\}$ Convergent. Since $\lim_{n\to\infty} \frac{1}{n^2} = 0$ = finite unique limit value

② $\{n\}$ divergent; Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$

= Infinite

③ $\{(-1)^n\}$ oscillates finitely.

Since $a_n = (-1)^n$

$$\underset{n\to\infty}{Lt}\; a_n = \underset{n\to\infty}{Lt}\; (-1)^n = \begin{cases} 1 & n = \text{even} \\ -1 & n = \text{odd} \end{cases}$$

④ $\{n^2(-1)^n\}$ oscillates infinitely.

because $$\lim_{n\to\infty} (-1)^n n^2 = \begin{cases} \infty & n = \text{even} \\ -\infty & n = \text{odd} \end{cases}$$

**Results** ① If $\{a_n\}$ converges to $l_1$ and $\{b_n\}$ converges to $l_2$ then

(i) $\{a_n + b_n\}$ converges to $l_1 + l_2$

(ii) $\{c a_n\} \longrightarrow c l_1$

(iii) $\{a_n b_n\} \longrightarrow l_1 l_2$

(iv) $\left\{\frac{a_n}{b_n}\right\} \longrightarrow \frac{l_1}{l_2}$ ; provided $l_2 \neq 0$.

R② Every convergent seq has a unique limit.

R③ Every convergent seq is bounded. but converse is not true

Exp ① $\{\frac{1}{n}\}$ is convergent seq. and it is bounded.

$a_n = \frac{1}{n} < 1$ for every $n$.

② For converse ie a bounded seq may not be convergent

$\{(-1)^n\}$ is bounded seq. but not convergent

$\therefore \lim_{n\to\infty} a_n = \begin{cases} 1 & n = \text{odd} \\ -1 & n = \text{even} \end{cases}$.

R④ Every convergent seq is bdd and has a unique limit.

R⑤ A Bounded monotonic seq is convergent

Exp $\{\frac{1}{n^2}\}$ is bounded since $\frac{1}{n^2} \le 1 \ \forall\ n \in N$. and Monotonically decreasing $a_{n+1} - a_n = \frac{1}{(n+1)^2} - \frac{1}{n^2}$

$$= \frac{\cancel{n^2} - \cancel{n^2} - 1 - 2n}{n^2(n+1)^2}$$

$$a_{n+1} - a_n \le 0$$

or $\boxed{a_{n+1} \le a_n}$

Hence, the seq is convergent, because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = \frac{1}{\infty} = 0$.

= finite.

**Some standard formula for limits**

① (i) $\lim_{n\to\infty} \frac{1}{n} = 0$, (ii) $\lim_{n\to\infty} \frac{1}{n^2} = 0$, (iii) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

② $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

③ $\lim_{n\to\infty} \frac{\log n}{n} = 0$

④ $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x$.

(5) $\underset{n\to\infty}{Lt}\ x^{1/n} = 1$, for $x > 0$.

(6) $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$

Question:- Determine the nature of following sequences whose $n^{th}$ term $a_n$ is

Exp (1) $a_n = \dfrac{n^2 - n}{2n^2 + n}$

Sol. $\underset{n\to\infty}{Lt}\ a_n = \underset{n\to\infty}{Lt}\left(\dfrac{n^2 - n}{2n^2 + n}\right) = \underset{n\to\infty}{Lt}\left(\dfrac{2n-1}{4n+1}\right)$ By L-Hospital Rule $\left(\dfrac{\infty}{\infty}\right)$

$= \underset{n\to\infty}{Lt}\left(\dfrac{2}{4}\right) = \dfrac{1}{2}$.

Seq. is Convergent since limit of seq. is unique & finite

Exp (2) $a_n = \tanh n$

Sol: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tanh n = \lim_{n\to\infty} \dfrac{\sinh n}{\cosh n} = \underset{n\to\infty}{Lt}\left(\dfrac{e^n - e^{-n}}{e^n + e^{-n}}\right)$

$= \lim_{n\to\infty} \dfrac{\cancel{e^{-n}}(e^{2n} - 1)}{\cancel{e^{-n}}(e^{2n} + 1)} = \underset{n\to\infty}{Lt} \dfrac{\cancel{e^{2n}}\left(1 - \frac{1}{e^{2n}}\right)}{\cancel{e^{2n}}\left(1 + \frac{1}{e^{2n}}\right)}$

$= 1 =$ unique & finite

So it is Convergent seq.

Exp (3) $a_n = e^n$

Sol $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^n = e^\infty = \infty$, so divergent.

Exp 4 $a_n = 2 + (-1)^n$

$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty}\left[2 + (-1)^{2n}\right] = 2 + 1 = 3.$

$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty}\left[2 + (-1)^{2n+1}\right] = 2 - 1 = 1.$

Sequence Oscillates finitely since it has more than one finite limits.

Exp 5

| (i) $\dfrac{2n+1}{1-3n}$ | (ii) $1 + \dfrac{(-1)^n}{n}$ | $\dfrac{1 + (-1)^n}{n}$ | $\sin n$ | $\dfrac{\log n}{n}$ | $\dfrac{1}{3^n}$ | $\dfrac{(n+1)^2}{n+1}$ |
|---|---|---|---|---|---|---|
| cgt, $l = -2/3$ | cgt, $l = 1$ | cgt $l = 0$ | dgt $l = \infty$ | cgt $l = 0$ | cgt $n = \frac{3}{2}$ | cgt |

**Infinite series :-** If $\langle u_n \rangle$ is a sequence of real numbers then the expression of the form

$$u_1 + u_2 + \cdots + u_n + \cdots\cdots$$

is called an infinite series and is denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$.

$u_n$ is the nth term of the series $\sum u_n$

eg. (1) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \cdots$

(2) $\sum_{n=1}^{\infty} \frac{x^n}{\lfloor n} = \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \cdots\cdots$

(3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$

**Series of +ve terms :-** If all terms of the series $\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots$ are +ve i.e. $u_n > 0 \;\forall\; n \in N$, then $\sum u_n$ is called series of +ve terms.

**Alternative Series :-** A series in which terms are alternatively +ve and −ve is called alternative series. Thus the series $\sum (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$ where $u_n > 0 \;\forall n \in N$, is an alternative series.

**Partial Sums**

If $\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots\cdots$ is an infinite series whose terms may be +ve or −ve, then we define a sequence $\langle S_n \rangle$ as follows

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$\vdots$$
$$S_n = u_1 + u_2 + \cdots + u_n = \sum_{r=1}^{n} u_r \text{ and so on.}$$

The seq $\langle S_n \rangle$ is called seq. of partial sums (SOPS) of the series $\sum u_n$.

**Behaviour of Infinite Series** An Infinite series $\sum u_n$ Converges or diverges or Oscillates ( finitely or infinitely) according as its SOPS $<S_n>$ Converges, diverges or oscillates (finitely or Infinitely).

Thus the series $\sum u_n$

(i) Converges if $\lim_{n\to\infty} S_n$ = finite limit value = S

Here S is the sum of the series

(ii) diverges if $\lim_{n\to\infty} S_n = \pm\infty$

(iii) Oscillates finitely if $\lim_{n\to\infty} S_n$ = more than one limit

(iv) Oscillates infinitely if $\lim_{n\to\infty} S_n = \pm\infty$

**Exp ①** $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+ \cdots\cdots$

Here $u_n = \frac{1}{4^{n-1}}$ so $\lim_{n\to\infty} S_n = \underset{n\to\infty}{Lt}\left[u_1+u_2+ \quad +u_n\right]$

Result

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\underset{n\to\infty}{Lt}\ x^n = 0 \ ; \quad x<1$$

$$= \underset{n\to\infty}{Lt}\left[1+\frac{1}{4}+\frac{1}{4^2}+ \quad +\frac{1}{4^{n-1}}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{1.\{1-\frac{1}{4^n}\}}{1-\frac{1}{4}}\right]$$

$$= \underset{n\to\infty}{Lt}\ \frac{4}{3}\left[1-\left(\frac{1}{4}\right)^n\right]$$

$$= \frac{4}{3}\left[1-0\right]$$

$$= \frac{4}{3} = \text{finite}$$

∴ Series Converges.

② $1^2+2^2+3^2+4^2+ \cdots +n^2+ \cdots\cdots$

Sol: Here $u_n = n^2$

$$S_n = u_1+u_2+ \quad +u_n^2$$

$$= 1^2+2^2+ \quad +n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} = \infty,$$ Series is divergent.

Exp 3 $7-4-3+7-4-3+7-4-3+ \text{———}$

Sol. Here $u_1 = 7$ $\quad u_5 = -4$
$u_2 = -4$ $\quad u_6 = -3$
$u_3 = -3$ $\quad u_8 = 7$
$u_4 = 7$

$\therefore$ $S_1 = u_1 = 7$

$S_2 = u_1 + u_2 = 7-4 = 3$

$S_3 = u_1 + u_2 + u_3 = 7-4-3 = 0$

$S_4 = u_1 + u_2 + u_3 + u_4 = 0+7 = 7$

$S_5 = u_1 + u_2 + u_3 + u_4 + u_5 = 7-4 = 3$

$S_6 = u_1 + u_2 + u_3 + u_4 + u_5 + u_6 = 3-3 = 0$

$$\therefore \lim_{n\to\infty} S_n = \begin{cases} 0 & \text{when } n = 3m \\ 7 & \text{,, } n = 3m+1 \\ 3 & \text{,, } n = 3m+2 \end{cases}$$

Since limit is not unique, series oscillates finitely.

Exp 4 $\sum u_n = \sum (-1)^n n = 1 + (-2) + 3 + (-4) + - - + (-1)^{n-1} n + -$

Sol: $u_n = (-1)^n n$.

$S_1 = 1$

$S_2 = 1-2 = 1$

$S_3 = 1-2+3 = 2$

$S_4 = 1-2+3-4 = -2$

$S_5 = -2+5 = 3$

$S_6 = 3-6 = -3$

$$\therefore S_n = \begin{cases} -\frac{n}{2} & n = \text{even} \\ \frac{n+1}{2} & n = \text{odd.} \end{cases}$$

$$\therefore \lim_{n\to\infty} S_n = -\infty \quad \text{; when } n = \text{even}$$
$$= +\infty \quad \text{when } n = \text{odd}$$

Series oscillates infinitely.

**Ex 5** Prove that the following series is convergent and find its sum.

$$\frac{2}{\lfloor 3} + \frac{3}{\lfloor 4} + \frac{4}{\lfloor 5} + \cdots \infty$$

**Sol:-** Here Take $u_n = \dfrac{n+1}{\lfloor n+2}$

$$\left[ \because \frac{n\text{th term of } 2,3,4\ldots}{\lfloor n\text{th term of } 3,4,5} = \frac{2+(n-1)\cdot 1}{\lfloor 3+(n-1)\cdot 1} = \frac{n+1}{\lfloor n+2} \right]$$

$$= \frac{(n+2)-1}{\lfloor n+2}$$

$$= \frac{(n+2)}{\lfloor n+2} - \frac{1}{\lfloor n+2}$$

$$= \frac{(n+2)}{(n+2)\lfloor n+1} - \frac{1}{\lfloor n+2}$$

$$u_n = \frac{1}{\lfloor n+1} - \frac{1}{\lfloor n+2}$$

$$S_n = u_1 + u_2 + \cdots + u_{n-1} + u_n$$

$$= \left(\frac{1}{\lfloor 2} - \cancel{\frac{1}{\lfloor 3}}\right) + \left(\cancel{\frac{1}{\lfloor 3}} - \cancel{\frac{1}{\lfloor 4}}\right) + \left(\cancel{\frac{1}{\lfloor 4}} - \frac{1}{\lfloor 5}\right) + \cdots + \left(\frac{1}{\lfloor n+1} - \cancel{\frac{1}{\lfloor n+1}}\right) + \left(\cancel{\frac{1}{\lfloor n+1}} - \frac{1}{\lfloor n+2}\right)$$

$$= \frac{1}{\lfloor 2} - \frac{1}{\lfloor n+2}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor n+2}\right) = \frac{1}{2}.$$

$\therefore$ $\Sigma u_n$ converges and its sum $= \frac{1}{2}$.

## Properties of Infinite Series

① If a series $\Sigma u_n$ converges to a sum $s$ then $\Sigma c u_n$ also converges to $cs$, where $c$ is constant.

② If $\Sigma u_n$ & $\Sigma v_n$ be two convergent series then $\Sigma(u_n+v_n)$ & $\Sigma(u_n-v_n)$ is also convergent.

③ If $\Sigma u_n$ is convergent & $\Sigma v_n$ divergent then $\Sigma(u_n+v_n)$ is divergent.

④. The nature of Infinite series does not change
(i) By Multiplying all terms by a constant $k$.
(ii) by addition or deletion of a finite number of terms

## Necessary Condition for Convergence of a Series

If a series $\sum u_n$ converges $\Rightarrow$ $\boxed{\lim_{n\to\infty} u_n = 0}$

**Proof**:- Let $S_n = \underbrace{u_1 + u_2 + \cdots + u_{n-1}} + u_n$.

$$S_n = S_{n-1} + u_n$$

$$\therefore \quad u_n = S_n - S_{n-1} \quad \text{---} (1)$$

Taking limit on both sides, we get

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \quad \text{---} (2)$$

We know that $\sum u_n$ is converges

$\Rightarrow$ $\langle S_n \rangle$ is converges and converges to K

$\Rightarrow$ $\lim_{n\to\infty} S_n = K$ & $\lim_{n\to\infty} S_{n-1} = K$.

By eq (2), we get

$$\boxed{\lim_{n\to\infty} u_n = K - K = 0}$$

**Note**. Converse of the above need not be true.

Exp (1) Series $\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \infty$

and $\lim_{n\to\infty} \frac{1}{n} = 0$. But $\sum \frac{1}{n}$ is not convergent.

**Remark**:- (1) $\sum u_n$ convergent $\Rightarrow$ $\lim_{n\to\infty} u_n = 0$.

(2) $\lim_{n\to\infty} u_n = 0 \Rightarrow \sum u_n$ may or may not be convergent

(3) $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \sum u_n$ is not convergent.

Exp (1) Test for convergence of the following series

(i) $1 + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{n}{n+1} + \cdots \infty$

(ii) $1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \cdots + \frac{2^n - 1}{2^n + 1} + \cdots \infty$

(iii) $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots + \infty$

**Sol**:- (i) We have $u_n = \frac{n}{n+1}$, $\lim_{n\to\infty} u_n = \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)$ $\left(\frac{\infty}{\infty}\right)$

$$= \underset{n\to\infty}{Lt} \frac{1}{1} = 1 \neq 0.$$

Hence given series is not convergent ie divergent

(ii) Here $u_n = \frac{2^n - 1}{2^n + 1}$.

$$\underset{n\to\infty}{Lt} u_n = \underset{n\to\infty}{Lt} \left(\frac{2^n - 1}{2^n + 1}\right) \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \underset{n\to\infty}{Lt} \frac{2^n \log 2}{2^n \log 2} = 1 \neq 0,$$ not convergent ie divergent

(iii) Here $u_n = \sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1+1/n}}$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1+1/n}} = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1+0}} = \frac{1}{\sqrt{2}} \neq 0.$$

$\therefore$ $\sum u_n$ is not convergent $\Rightarrow$ divergent.

Que/ Test for convergence of series $\sum \cos\frac{1}{n}$, $\sum \cos\frac{1}{n^2}$

Ans Not convergent.

## Standard Infinite series

### ① Geometric series

The series $1 + r + r^2 + \cdots \infty$ is

(i) convergent if $|r| < 1$

(ii) divergent if $r \geq 1$

(iii) oscillates if $r \leq -1$.

Proof:- We have

$$S_n = 1 + r + r^2 + \quad + r^{n-1}$$

$$= \frac{1 - r^n}{(1 - r)}$$

$$= \frac{1}{(1-r)} - \frac{r^n}{(1-r)}.$$

Sum of n term of G.P.

$$S_n = \frac{a(1-r^n)}{(1-r)}, r<1$$

$$= \frac{a(r^n - 1)}{(r-1)}, r>1.$$

(i) when $|r|<1$

$$\underset{n\to\infty}{Lt} S_n = \frac{1}{1-r} - \frac{1}{(1-r)} \underset{n\to\infty}{Lt} r^n \qquad \left(\lim_{n\to\infty} r^n = 0, \text{ if } |r|<1\right)$$

$$= \frac{1}{(1-r)}$$

$\Rightarrow$ $\langle S_n \rangle$ is Convergent

$\Rightarrow$ The given series is Convergent

(ii) When $r \geq 1$

Subcase-I When $r = 1$

$$S_n = 1 + 1 + \cdots + 1 \text{ (n times)} = n$$

$$\boxed{\lim_{n \to \infty} S_n = \infty}$$

$\Rightarrow$ $\langle S_n \rangle$ diverges to $+\infty$

$\Rightarrow$ given series diverges.

Subcase-II. When $r > 1$

$\because \lim_{n \to \infty} r^n = \infty$ ; $r > 1$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( \frac{r^n - 1}{r - 1} \right) = \infty$$

$\Rightarrow$ The given series is divergent

(iii) When $r \leq -1$

Subcase-I When $r = -1$ series becomes $1 - 1 + 1 - 1 + 1 - 1 \cdots \infty$

$$S_n = \begin{cases} 0 & \text{when } n = \text{even} \\ 1 & \text{when } n = \text{odd} \end{cases}$$

$\therefore$ $S_{2n-1} \to 1$ and $S_{2n} \to 0$

$\Rightarrow$ $\langle S_n \rangle$ oscillates finitely.

$\Rightarrow$ Series oscillates finitely.

Subcase-II When $r < -1$, let $r = -k$, where $k > 1$.

$$\therefore r^n = (-k)^n = (-1)^n k^n.$$

$$\therefore \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^n}{(1 - r)} = \lim_{n \to \infty} \frac{1 - (-1)^n k^n}{1 + k}$$

$= \infty$ if $n = \text{odd}$

$= -\infty$ if $n = \text{even}$

$\Rightarrow$ Series oscillates infinitely.

**Note:-** Geometrical series converges only when its common ratio is numerically less than 1.

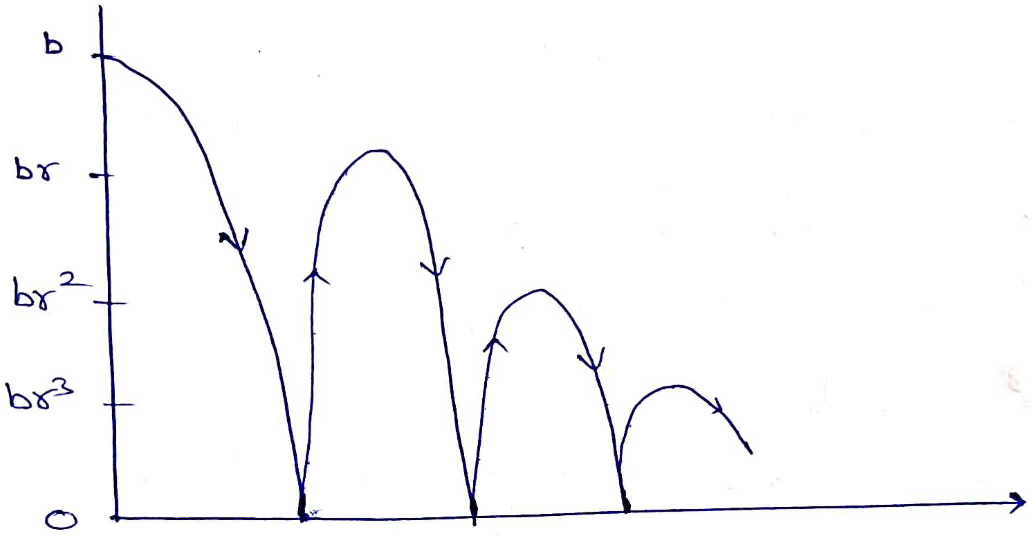
eg (i) $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is cgt $(\because r = \frac{1}{2} < 1)$

(ii) $\sum a_n = 1 + 1 + 1 + 1 + \cdots$ is dgt $(\because r = 1)$

(iii) $\sum 3^n = 3 + 3^2 + 3^3 + \cdots$ is dgt $(\because r = 3 > 1)$

V.V.Imp

Exp 1:- A ball is dropped from a height b feet from a flat surface. Each time the ball hits the ground after falling a distance h it rebonds a distance rh where $0 < r < 1$.




Find the total distance the ball travels if $b = 4$ ft and $r = 3/4$.

Sol. The total distance travelled by the ball is given by infinite Geometric Series

$$S = b + 2br + 2br^2 + 2br^3 + \cdots$$

$$= b + 2br(1 + r + r^2 + \cdots)$$

$$= b + 2br\left(\frac{1}{1-r}\right) = \frac{(1-r)b + 2br}{(1-r)} = \frac{b(1+r)}{(1-r)}.$$

For $b = 4$ & $r = \frac{3}{4}$. The distance $= 4\left(\frac{1+3/4}{1-3/4}\right) = 28$ ft.

Ans

V.V. Imp

## (2) Hormonic Series of order p or p-Harmonic series or p-Series test

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \infty$

is (i) Convergent if $p > 1$

and (ii) divergent if $p \le 1$.

**Cauchy's Integral Test**

If $u(x)$ is a +ve monotonically decreasing fuction such that $u(n) = u_n \ \forall \ n \in N$, Then $\sum_{n=1}^{\infty} u_n$ and $\int_1^{\infty} u(x)\,dx$ Converge or diverge together.

**Proof:–** To Apply Cauchy's Integral test, Consider $u(x) = \frac{1}{x^p}$

Then

$$\int_1^{\infty} u(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx$$

$$= \lim_{t \to \infty} \int_1^{t} x^{-p}\,dx \qquad \text{——— (1)}$$

$$= \lim_{t \to \infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^t$$

$$= \frac{1}{(1-p)} \lim_{t \to \infty} \left[ (t)^{1-p} - 1 \right]$$

$$= \frac{1}{(1-p)} \begin{cases} (0-1) & \text{if } p>1 \\ \infty & \text{if } p<1 \end{cases}$$

$$\because \lim_{t \to \infty} t^{1-p} = 0 \quad p>1$$
$$= \infty \quad p<1$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1. \end{cases}$$

**Also for $p=1$**, from (1)

$$\int_1^{\infty} u(x)\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t \to \infty} (\log t - \log 1) = \log \infty$$

$$= \infty.$$

$$\therefore \int_1^{\infty} u(x)\,dx = \begin{cases} \frac{1}{p-1} = \text{finite} & p>1 \\ \infty = \text{Infinite} & p<1 \\ \infty = \text{Infinite}, & p=1 \end{cases}$$

Thus $p$-harmonic series converges when $p > 1$ and diverges when $p \le 1$.

③ **Comparison test** (For Positive term series)

ⓐ **First Comparison test** :- If $\sum u_n$ & $\sum v_n$ be two +ve term series such that $u_n \le v_n \quad \forall\ n \in N$.

Then

(i) $\sum v_n$ converges $\Rightarrow$ $\sum u_n$ converges

(ii) $\sum u_n$ diverges $\Rightarrow$ $\sum v_n$ diverges.

V.V. Imp

ⓑ **Limit form Comparison test** :- If $\sum u_n$ & $\sum v_n$ be two +ve term series such that

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = l \quad (l \text{ is finite \& non-zero})$$

Then $\sum u_n$ & $\sum v_n$ behave alike.

**Remark** If $l = 0$ or $\infty$, then conclusion of above test may not hold good.

**Exp ①** Test convergence of the following series.

(i) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots + \frac{1}{n^n} + \cdots$

(ii) $\sum e^{-n^2}$

(iii) $\sum \frac{1}{\lfloor n}$

(iv) $\sum \frac{1}{n\,2^n}$

(v) $\sum \frac{1}{\sqrt{n}}$

**Sol**:- (i) clearly $n^n > 2^n$ for $n > 2$

$$\Rightarrow \frac{1}{n^n} < \frac{1}{2^n} \text{ for } n > 2$$

Since $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is a Geometrical series with common ratio $r = \frac{1}{2} < 1$ so $\sum \frac{1}{2^n}$ is convergent. Hence By first Comparison test $\sum \frac{1}{n^n}$ is convergent.

(ii) We have $e^x > x$ for $x > 0$.

Then $e^{n^2} > n^2$

$$\Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2} \Rightarrow e^{-n^2} < \frac{1}{n^2} \quad \forall n.$$

Since $\sum \frac{1}{n^2}$ = Convergent (here $p = 2 > 1$) so By first Comparison test $\sum e^{-n^2}$ is Convergent.

(iii) We have $\lfloor n = 1.2.3.4.5 \text{———} n$

$$\geq 1.2.2.2 \text{- - - - - - -}$$

$$= 2^{n-1}$$

$$\therefore \lfloor n \geq 2^{n-1} \quad \forall n \geq 2.$$

$$\Rightarrow \frac{1}{\lfloor n} \leq \frac{1}{2^{n-1}}$$

Now $\sum \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \text{- -}$ being a Geometrical series with common ratio $r = \frac{1}{2} < 1$, is convergent so by first Comparison test $\sum \frac{1}{\lfloor n}$ is convergent.

(iv) Here $n\,2^n \geq 2^n \Rightarrow \frac{1}{n\,2^n} \leq \frac{1}{2^n}$

For $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \text{- -}$ Geometrical series with $r = \frac{1}{2} < 1$, is Convergent so by first comparison test $\sum \frac{1}{n\,2^n}$ is cgt.

(v) $\sum u_n = \sum \frac{1}{\sqrt{n}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \text{- - -} + \text{- - —}$ Here $p = \frac{1}{2} < 1$

So By p-series test it is divergent.

Exp (2) Test Convergence of the following

(i) $\sum_{n=1}^{\infty} \left(\frac{2^n+3}{3^n+1}\right)^{1/2}$ (ii) $\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$

(iii) $\sum \left(\sqrt[3]{n^3+1} - n\right)$ (iv) $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p (b+n)^q}$, where $a, b, p, q$ are +ve constants

Sol:- (i) Here $u_n = \left(\frac{2^n+3}{3^n+1}\right)^{1/2}$ ; $v_n = \left(\frac{2^n}{3^n}\right)^{1/2}$

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \sqrt{\frac{2^n+3}{3^n+1}} \cdot \sqrt{\frac{3^n}{2^n}} = \lim_{n\to\infty} \sqrt{\frac{(2^n+3)\,3^n}{(3^n+1)\,2^n}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1+\frac{3}{2^n}}{1+\frac{1}{3^n}}} = 1 = \text{Non-zero \& finite.}$$ (18)

$\therefore$ both series $\sum u_n$ & $\sum v_n$ behave alike.

But $\sum v_n = \sum \left(\sqrt{\frac{2}{3}}\right)^n = \sqrt{\frac{2}{3}} + \left(\sqrt{\frac{2}{3}}\right)^2 + \cdots$ is Geometrical series with common ratio $r = \sqrt{2/3} < 1$, is Convergent.

So $\sum u_n$ is also convergent (By limit form test).

(ii) $$u_n = \frac{\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)}{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)} \cdot \frac{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}{}$$

$$= \frac{n^4+1-(n^4-1)}{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)} = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}} \sim \frac{2}{n^2+n^2} \sim \frac{1}{n^2}$$

Take $v_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left[\frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}} \cdot \frac{n^2}{1}\right]$$

$$= \lim_{n\to\infty} \frac{2\cancel{n^2}}{\cancel{n^2}\left[\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}\right]} = \frac{2}{2} = 1 \neq 0 \text{ \& finite}$$

So By limit form comparison test $\sum u_n$ & $\sum v_n$ behave alike.

But $\sum \frac{1}{n^2} = \sum v_n$ is Convergent by p-series test.

$\therefore$ $\sum u_n$ is Convergent.

(iii) $$u_n = (n^3+1)^{1/3} - n \qquad \left| \because (1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \cdots \right.$$

$$= n\left[1+\frac{1}{n^3}\right]^{1/3} - n$$

$$= n\left[1 + \frac{1}{3}\frac{1}{n^3} + \frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)}{\lfloor 2}\frac{1}{n^6} + \cdots\right] - n$$

$$= n\left[\cancel{1} + \frac{1}{3}\frac{1}{n^3} - \frac{1}{9}\frac{1}{n^6} + \cdots - \cancel{1}\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3} - \frac{1}{9}\frac{1}{n^3} + \cdots\right] = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9}\frac{1}{n^3} + \cdots\right]$$

Let $v_n = \frac{1}{n^2}$.

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt} \frac{\frac{1}{n^2}\left(\frac{1}{3}-\frac{1}{9n^3}+\cdots\right)}{\frac{1}{n^2}} = \underset{n\to\infty}{Lt}\left(\frac{1}{3}-\frac{1}{9n^3}+\cdots\right) = \frac{1}{3}$$

= Non zero and finite.

$\therefore$ By limit form Comparison test $\Sigma u_n$ & $\Sigma v$ behave alike.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ ; Convergent by p-series test. So given series $\Sigma u_n$ is Convergent.

(iv) $u_n = \dfrac{1}{(a+n)^p (b+n)^q} \sim \dfrac{1}{n^p \cdot n^q} = \dfrac{1}{n^{p+q}}$. Take $v_n = \dfrac{1}{n^{p+q}}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{1}{(a+n)^p (b+n)^q} \cdot n^{p+q}$$

$$= \underset{n\to\infty}{Lt} \frac{\cancel{n^{p+q}}}{\cancel{n^{p+q}}\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q} = \frac{1}{1} = 1$$

= Non zero and finite.

$\therefore$ By limit form Comparison test, $\Sigma u_n$ & $\Sigma v_n$ behave alike

But $\Sigma v_n = \Sigma \frac{1}{n^{p+q}}$ is Convergent if $p+q>1$ & divergent if $(p+q) \le 1$.

So Given $\Sigma u_n$ is Convergent if $p+q>1$ & divergent if $p+q \le 1$.

**Exp (3)** Test the series $\sum_{n=1}^{\infty} \frac{1}{n+10}$ for Convergence or divergence

Sol: Here $u_n = \frac{1}{n+10}$, $v_n = \frac{1}{n}$.

$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{n+10} \cdot n\right) = 1$. = Non zero & finite so by limit form test given series $\Sigma u_n$ & $\Sigma v_n$ behave alike. But $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent by p-series test. So $\Sigma u_n$ is divergent.

**Exp 4** Test for Convergence of the series $\frac{1}{3\cdot 7} + \frac{1}{4\cdot 9} + \frac{1}{5\cdot 11} + \cdots$

Sol: We have $u_n = \dfrac{1}{(n+2)(2n+5)} \sim \dfrac{1}{n \cdot n} = \dfrac{1}{n^2}$

Take $v_n = \frac{1}{n^2}$.

$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt} \frac{1}{(n+2)(2n+5)} \cdot n^2 = 1$ = Non zero & finite,

**D'Alembert's Ratio test :-** If $\sum u_n$ is a +ve term series such that $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = l$. Then $\sum u_n$ is

(a) Convergent if $l<1$
(b) divergent if $l>1$
(c) Test fails for $l=1$.

**Que 1)** Test for Convergence the series $\frac{1}{1.2}+\frac{1}{2.2^2}+\frac{1}{3.2^3}+\frac{1}{4.2^4}+\ldots$

**Sol:** We have $u_n = \frac{1}{n.2^n}$, $u_{n+1} = \frac{1}{(n+1)\,2^{n+1}}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1}{(n+1)(2^{n+1})}\cdot\frac{n.2^n}{}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{2^n}{2^{n+1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)\cdot\left(\frac{1}{2}\right) = \frac{1}{2} < 1.$$

So By D'Alembert's Ratio test given series is convergent

**Que 2)** test for Convergence of the series whose nth term is

$$\frac{(n+1)!}{3!\;n!\;3^n}.$$

**Sol:** $u_n = \frac{\lfloor n+1}{\lfloor 3\;\lfloor n\;3^n}$, $u_{n+1} = \frac{\lfloor n+2}{\lfloor 3\;\lfloor n+1\;3^{n+1}}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{\lfloor n+2}{\cancel{\lfloor 3}\;\lfloor n+1\;3^{n+1}}\cdot\frac{\cancel{\lfloor 3}\;\lfloor n\;3^n}{\lfloor n+1}$$

$$= \frac{(n+2)\,\cancel{\lfloor n+1}}{(n+1)\,\cancel{\lfloor n}\;\cancel{3^n}.3}\cdot\frac{\cancel{\lfloor n}\;\cancel{3^n}}{\cancel{\lfloor n+1}} = \frac{1}{3}\left(\frac{n+2}{n+1}\right)$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+2}{n+1}\right) = \lim_{n\to\infty}\frac{1}{3}\left(\frac{1+2/n}{1+1/n}\right) = \frac{1}{3}<1$$

So By D'Alembert Ratio test given series is cgt.

Que 3) Discuss the Convergence of the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n \quad (x>0)$.

Sol: Here we have $u_n = \sqrt{\frac{n}{n^2+1}}\, x^n$, $u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}} \cdot x^{n+1}$.

$$\frac{u_{n+1}}{u_n} = \sqrt{\frac{n+1}{(n+1)^2+1}}\, x^{n+1} \cdot \sqrt{\frac{n^2+1}{n}}\, \frac{1}{x^n}$$

$$= \sqrt{\frac{(n+1)(n^2+1)}{(n^2+2n+2)\,n}}\, x = \sqrt{\frac{n^3(1+\frac{1}{n})(1+\frac{1}{n^2})}{n^3(1+\frac{2}{n}+\frac{2}{n^2})}}\, x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \sqrt{\frac{(1+\frac{1}{n})(1+1/n^2)}{(1+\frac{2}{n}+\frac{2}{n^2})}}\, x = x.$$

By D'Alembert's Ratio test, given $\sum u_n$ converges if $x<1$ diverges if $x>1$ and test fails if $x=1$.

When $x=1$. $u_n = \sqrt{\frac{n}{n^2+1}} \sim \sqrt{\frac{n}{n^2}} = \sqrt{\frac{1}{n}}$. So Take $v_n = \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \sqrt{\frac{n}{n^2+1}} \cdot \sqrt{n} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2(1+1/n^2)}} = 1 \neq 0 \text{ \& finite}$$

So By Limit form Comparison test $\sum u_n$ & $\sum v_n$ behave a like but $\sum v_n = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ is divergent by p-series test.

$\therefore \sum u_n$ is divergent. for $x=1$

Hence the given series $\sum u_n$ is Convergent if $x<1$ and divergent when $x \geq 1$.

Que 4) Discuss the Convergence of the series whose nth term is

$$\frac{(1+\alpha)(1+2\alpha)\cdots(1+n\alpha)}{(1+\beta)(1+2\beta)\cdots(1+n\beta)}$$

Ans: Converges for $\beta > \alpha$ and diverges $\alpha \geq \beta$.

## Raabe's test (or Higher Ratio test)

If $\sum u_n$ is a series of +ve terms such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = l, \text{ then } \sum u_n \text{ is}$$

(a) Convergent if $l > 1$.
(b) divergent if $l < 1$
(c) test fails for $l = 1$.

**Note:**– Raabe's test is stronger than D' Alembert ratio test and is applied only when D' Alembert's ratio test fails.

**Que 1)** Test the Convergence of the series $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+\cdots$

**Sol:**– Here $u_n = \frac{x^n}{(2n-1)(2n)}$, $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$.

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(2n+1)(2n+2)} \cdot \frac{(2n-1)(2n)}{x^n} = \frac{(2n-1)(2n)}{(2n+1)(2n+2)}x.$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \frac{(2n-1)(2n)}{(2n+1)(2n+2)}x$$

$$= \lim_{n\to\infty} \frac{\cancel{2n}\left(1-\frac{1}{2n}\right)\cancel{2n}}{\cancel{2n}\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)\cancel{2n}}x = x \quad (1)$$

By D' Alembert's Ratio test $\sum u_n$ is convergent if $x<1$ and divergent if $x>1$.

Test fails when $x=1$

Let us apply Raabe's test when $x=1$,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{(2n-1)(2n)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{4n^2}+6n+2-\cancel{4n^2}+2n}{(2n-1)(2n)}\right]$$

$$= \lim_{n\to\infty} \frac{n^2(8+2/n)}{n^2(2-1/n)2} = 2 > 1.$$

So By Raabe's test, given series $\sum u_n$ is convergent

Hence, we can say $\sum u_n$ is convergent if $x \leq 1$ and divergent if $x > 1$.

Que 2) Test for convergence or divergence of the series

$$1 + \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \, 6} + \cdots$$

Sol: Here $u_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \, 6 \cdots (2n)}$.

$$u_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots}{2 \cdot 4 \, 6 \cdots} \frac{(2n-1)\,(2n+1)}{(2n)\,(2n+2)}.$$

$$\lim_{n\to\infty} \left( \frac{u_{n+1}}{u_n} \right) = \lim_{n\to\infty} \frac{\cancel{1 \cdot 3 \cdot 5 \cdots (2n-1)}\,(2n+1)}{\cancel{2 \cdot 4 \, 6 \cdots (2n)}\,(2n+2)} \cdot \frac{\cancel{2 \, 4 \, 6 \cdots (2n)}}{\cancel{1 \cdot 3 \cdot 5 \cdots (2n-1)}}$$

$$= \lim_{n\to\infty} \frac{(2n+1)}{(2n+2)} = \lim_{n\to\infty} \frac{\cancel{2n}\left(1 + \frac{1}{2n}\right)}{\cancel{2n}\left(1 + \frac{1}{n}\right)} = 1$$

$\therefore$ D'Alembert Ratio test fails. Now Applying Raabe's test

$$\lim_{n\to\infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n\to\infty} n \left[ \frac{2n+2}{2n+1} - 1 \right]$$

$$= \lim_{n\to\infty} n \left[ \frac{\cancel{2n}+2 - \cancel{2n} - 1}{2n+1} \right] = \lim_{n\to\infty} \left( \frac{n}{2n+1} \right)$$

$$= \frac{1}{2} < 1$$

So By Raabe's test given series is convergent.

Que 3) Discuss the convergence of the series

$$\frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \, 6} \cdot \frac{x^7}{7} + \cdots \quad (x > 0)$$

Sol: Here neglecting the first term, we have

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \, 4 \, 6 \cdots (2n)} \cdot \frac{x^{2n+1}}{(2n+1)}.$$

$$u_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\,(2n+1)}{2 \cdot 4 \cdot 6 \cdot (2n)\,(2n+2)} \cdot \frac{x^{2n+3}}{(2n+3)}.$$

$$\frac{u_{n+1}}{u_n} = \frac{\cancel{1\cdot3\cdot5\cdots(2n-1)}(2n+1)}{\cancel{2\cdot4\cdot6\cdots(2n)}(2n+2)} \frac{x^{2n+3}}{(2n+3)} \cdot \frac{\cancel{2\cdot4\cdot6\cdots(2n)}}{\cancel{1\cdot3\cdot5\cdots(2n-1)}} \frac{(2n+1)}{x^{2n+1}}$$

$$= \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)} x^2.$$

$$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt} \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)} x^2$$

$$= \underset{n\to\infty}{Lt} \frac{2n\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{2n}\right)\cdot\cancel{2n}}{2n\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)\cancel{2n}} x^2 = x^2$$

By Ratio test (or D'Alembert's Ratio test), given series is Convergent if $x^2<1$ and divergent if $x^2>1$.
Thus test fails when $x^2=1$. So now applying Raabe's test when $x^2=1$.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \underset{n\to\infty}{Lt}\, n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right]$$

$$= \underset{n\to\infty}{Lt}\, n\left[\frac{\cancel{4n^2}+6n+4n+6-\cancel{4n^2}-1-\cancel{4n}}{4n^2+1+4n}\right]$$

$$= \underset{n\to\infty}{Lt}\, \cancel{n^2}\left[\frac{6+5/n}{\cancel{n^2}\left(4+\frac{1}{n}+\frac{1}{n^2}\right)}\right]$$

$$= \frac{6}{4} = \frac{3}{2} > 1.$$

By Raabe's test given series is Convergent.

Hence, $\sum u_n$ is Convergent when $x^2\le1$ and divergent when $x^2>1$.

**Que 4]** Test for Convergence of the series

$$1 + a + \frac{a(a+1)}{1\cdot2} + \frac{a(a+1)(a+2)}{1\cdot2\cdot3} + \cdots$$

**Ans** converges for $a\le0$ and diverges for $a>0$.

Ques 5) Test the following series for Convergence (25)

$$\frac{1}{2}x + x^2 + \frac{9}{8}x^3 + x^4 + \frac{25}{32}x^5 + \cdots \infty$$

Sol: Here $u_n = \frac{n^2 \cdot x^n}{2^n}$.

Ans cgt if $x < 2$
dgt if $x \geq 2$.

## Cauchy's nth root test

If $\Sigma u_n$ is a +ve term series such that $\lim_{n\to\infty} (u_n)^{1/n} = l$. Then $\Sigma u_n$ is

(a) Convergent if $l < 1$,
(b) divergent if $l > 1$.
(c) Test fails if $l = 1$..

Note: It is applicable when $u_n$ involves the nth power of itself as a whole.

Ques 1) Test Convergence of $\Sigma \left(\frac{n+1}{2n+5}\right)^n$.

Sol:. Here $u_n = \left(\frac{n+1}{2n+5}\right)^n$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n+1}{2n+5}\right)^n\right]^{1/n} = \lim_{n\to\infty}\left(\frac{n+1}{2n+5}\right) = \frac{1}{2} < 1$$

So By Cauchy's nth root test given series is Convergent.

Ques 2) $\Sigma \frac{(1+\frac{1}{n})^{2n}}{e^n}$

Sol:— $u_n = \frac{(1+\frac{1}{n})^{2n}}{e^n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})^{2n}}{e^n}\right]^{1/n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{e} = \frac{1}{e} = 0.3678 < 1.$$

So By Cauchy's nth root test given series is Convergent.

Que 3) $\sum \frac{x^{2n}}{2^n}$ ; $x>0$.

Sol:- $u_n = \frac{x^{2n}}{2^n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{x^{2n}}{2^n}\right)^{1/n} = \lim_{n\to\infty} \frac{x^2}{2} = \frac{x^2}{2}.$$

By Cauchy's nth root test given series is convergent if $\frac{x^2}{2} < 1$ i.e. $x < \sqrt{2} = 1.414$ and divergent if $\frac{x^2}{2} > 1$ i.e. $x > \sqrt{2}$.

When $\frac{x^2}{2} = 1$ then test fails.

When $x = \sqrt{2}$, $u_n = 1$ for all $n$.

so $\lim_{n\to\infty} u_n = 1 \neq 0$,

So series is divergent.

Hence given series is convergent if $x < \sqrt{2}$ and divergent if $x \geq \sqrt{2}$.