Unit-I (Differential Equations)

Definition:- An Equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called differential Equation.

Examples

1. $y + \sin x = e^x$ (Not differential Equation)

2. $\frac{dy}{dx} = x + \sin x$ ——— (1)

3. $\frac{d^4x}{dt^4} + \left(\frac{dx}{dt}\right) + \left(\frac{d^2x}{dt^2}\right) = e^t$ ——— (2)

4. $y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{\left(\frac{dy}{dx}\right)}$ ——— (3)

5. $\rho \cdot \frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$ ——— (4)

6. $\frac{\partial^2 V}{\partial t^2} = k \cdot \left(\frac{\partial^3 V}{\partial x^3}\right)^2$ ——— (5)

7. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (Laplace Equation) ——— (6)

8. $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)$ ——— (7) (~~Heat~~ wave Eq)

9. $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)$ ——— (8) (Heat Eq.)

Kinds of differential Equation

O.D.E (Ordinary diff. Equation)

Def:- A differential Equation involving derivatives of one or more dependent variables w.r.to the single independent variable, is called O.D.E.

Examples (2), (3), (4) & (5).

P.D.E (Partial differential Equation)

Def. A diff Eq. involving partial derivatives of one or more dependent variables with respect to more than one independent varibles

Examples (6), (7) (8) and (9).

Order :- The order of the differential Equation is the order of highest order derivative involving in the differential Equation

Degree The degree of the differential Equation is the power of the highest order derivative involving in the differential Equation when the diff. Equation is free from radical signs and fractional powers.

Exp: (2) having order = 1, D=1.

(3) " O = 4, D=1.

(4) can be put in the form $y\left(\frac{dy}{dx}\right) = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + K.$

$\therefore$ $\boxed{O=1}$ $\boxed{D=2}$

(5) $\boxed{O=2}$ for degree, we first remove all fractional powers so squaring on both sides, we get

$$\rho^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

$\therefore$ $\boxed{\text{Degree} = 2}$

(6) $\boxed{O=3}$ & $\boxed{D=2}$

(7) $\boxed{O=2}$ & $\boxed{D=1}$

(8) $\boxed{O=2}$ & $D=1$

(9) $\boxed{O=2}$ & $\boxed{D=1}$

(10) Find order and degree of $\left(\frac{d^3y}{dx^3}\right)^{2/3} + \left(\frac{d^3y}{dx^3}\right)^{3/2} = 0.$

Sol: $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}\times 6} + \left(\frac{d^3y}{dx^3}\right)^{\frac{3}{2}\times 6} = 0 \Rightarrow \left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{d^3y}{dx^3}\right)^9 = 0$

$\therefore$ $\boxed{O=3}$ and $\boxed{D=9}$

(11) Find order and degree of $\sin\left(\frac{d^2y}{dx^2}\right) = y-x$

Sol:. $\boxed{O=2}$ for degree we expand it as

$$\left(\frac{d^2y}{dx^2}\right) - \frac{1}{\underline{|3}}\left(\frac{d^2y}{dx^2}\right)^3 + \frac{1}{\underline{|5}}\left(\frac{d^2y}{dx^2}\right)^5 - - - = y-x$$

So degree of above problem does not exist.

Remark:- The order of diff. Equation always exists but degree may or may not be exists.

(12) Find the order and degree of $\frac{d^2y}{dx^2} + \iint y\, dx\, dx = 0$

Sol:- It is not diff Eq. so we first convert it into diff Eq. so diff. it two times w.r.to x, we get

$$\frac{d^3y}{dx^3} + \int y\, dx = 0$$

$$\boxed{\frac{d^4y}{dx^4} + y = 0}$$

Order = 4 & Degree = 1.

(13) Find the order of diff Eq. $\frac{d^2y}{dx^2} + \iint \phi(x)\, dx\, dx = 0$. Also find degree. Here $\phi(x) =$ is function of x.

Sol: Let us take $\phi(x) = x$. So given Eq becomes

$$\frac{d^2y}{dx^2} + \iint x\, dx\, dx = 0$$

$$\frac{d^2y}{dx^2} + \int \frac{x^2}{2}\, dx = 0$$

$$\frac{d^2y}{dx^2} + \frac{x^3}{6} = 0$$

$\boxed{\text{Order} = 2}$ & $\boxed{\text{degree} = 1}$

Linear and Non-linear differential Equation

A differential Equation is said to be linear differential Equation if

(i) dependent Variable y and it's various derivatives occurs in the first degree only

(ii) They are not multiplied together and

(iii) not containing the transcendental function of y.

A diff Eq. which is not linear is called Non-linear.

Exp (1) $x^3 y''' + \sin x\, y'' + e^x y = x^3 e^x$ (L)

(2) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = e^x$ (NL)

(3) $y'' + y' + y = \sin y + e^y$ (NL)

Solution of diff. Equation:- A solution of diff Eq. is a relation between dependent variable and independent varibles when it is subtituted in the diff Eq. then diff Eq. becomes to Identity.

Exp $y = ce^{2x}$ is the solution of $\frac{dy}{dx} = 2y$, because it is put in the diff Eq. then diff Eq. becomes to identity.

Types of Solutions

General Solution — Particular solution.

Def A sol. of diff Eq. is called general solution if no. of arbitrary constants is equal to order of diff Eq.

Exp $y = \frac{A}{x} + B$ is the general solution of $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$, where A and B are arbitrary constants.

Def (Particular Solution):- A particular solution of diff Eq. is that solution which is obtained from general solution by giving particular values of arbitrary constants.

Exp Putting $A=2$ & $B=3$ then $y = \frac{2}{x}+3$ is the particular sol. of

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0.$$

Linear differential Equation of nth order with Constant coefficients

An Equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = X \quad \text{——— (1)}$$

where $a_0 \neq 0, a_1, a_2 \text{———} a_{n-1}, a_n$ are constants and X is either constant or function of x only, is called linear diff Eq. of nth order with constant coefficients.

Now, if we put $D \equiv \frac{d}{dx}$, then (1) becomes

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)y = X$$

or $\boxed{f(D)y = X}$ ——— (2)

where $f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$.

Solution of Equation (2) is $y =$ C. F. + P. I.

= Complementary function + Particular Integral

To find complementary function :- A.E. is $f(D)=0$ when $D=m$

i.e. $f(m)=0$

$$a_0 m^n + a_1 m^{n-1} + \text{------} + a_{n-1} m + a_n = 0$$

This Equation gives n roots say $m_1, m_2, \text{------} m_n$.

Case-I When all roots are real and distinct i.e. $m_1, m_2, \text{---} m_n$

$$\boxed{C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \text{------} + c_n e^{m_n x}}$$

If $m_1 = m_2 = m_3 = m$, then

$$C.F = (c_1 + x c_2 + x^2 c_3) e^{mx} + c_4 e^{m_4 x} + \text{---} + c_n e^{m_n x}.$$

Case-(II) Let $\alpha \pm i\beta$ pair of complex roots

$$\boxed{C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]}$$

or $= c_1 e^{\alpha x} \cos(\beta x + c_2)$ or $= c_1 e^{\alpha x} \sin(\beta x + c_2)$

Repeated complex roots $\alpha \pm i\beta$, $\alpha \pm i\beta$ (two times)

$$C.F = e^{\alpha x}[(c_1 + x c_2)\cos\beta x + (c_3 + x c_4)\sin\beta x].$$

Case-(III) Pair of surd or irrational roots i.e. $\alpha \pm \sqrt{\beta}$

$$\boxed{C.F = e^{\alpha x}[c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x]}$$

Que 1) Solve $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0.$

Sol:- A.E. is $m^3 - 7m - 6 = 0$ ——— (1)

Put $m=1$, $1-7-6 = -12 \neq 0$

Put $m=-1$ $-1+7-6=0$

$\therefore (m+1)$ is the factor of the eq (1).

$\therefore m^2(m+1) - m(m+1) - 6(m+1) = 0$

$(m+1)(m^2 - m - 6) = 0$

$m+1=0$, $m^2 - m - 6 = 0$

$\boxed{m=-1}$ $m^2 - (3-2)m - 6 = 0$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$(m-3)(m+2) = 0$

$\therefore m = -2, 3$

We get $m = -1, -2, 3$ (All roots are real and distinct)

C.F $= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$.

P.I. $= 0$ (Because R.H.S. of given diff Eq is zero).

So complete solution is

$y =$ C.F + P.I

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}}$$ Ans

Que 2) Solve the diff Eq $(D^2+1)^3 (D^2+D+1)^2 y = 0$. where $D = \frac{d}{dx}$.

Sol:- A.E. is $(m^2+1)^3 (m^2+m+1)^2 = 0$.

$(m^2+1)^3 = 0$

$(m^2+1)(m^2+1)(m^2+1) = 0$

$m^2+1 = 0$.

$m^2 = -1$

$m = 0 \pm i$

$m = 0 \pm i, 0 \pm i, 0 \pm i$

$= \alpha \pm i\beta$

& $(m^2+m+1)^2 = 0$

$(m^2+m+1) = 0$

$m = \frac{-1 \pm \sqrt{1^2 - 4\times1\times1}}{2\times1} = \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

$m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

We get roots

$m = 0 \pm i, 0 \pm i, 0 \pm i,$ $(\alpha = 0, \beta = 1)$ $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. $(\alpha = -\frac{1}{2}$ & $\beta = \frac{\sqrt{3}}{2})$

$$\text{C.F.} = e^{0x}\left[(c_1 + xc_2 + x^2c_3)\cos x + (c_4 + xc_5 + x^2c_6)\sin x\right] + e^{-\frac{1}{2}x}\left[(c_7 + xc_8)\cos\left(\tfrac{\sqrt{3}}{2}x\right) + (c_9 + xc_{10})\sin\left(\tfrac{\sqrt{3}}{2}x\right)\right].$$

P.I. $= 0$.

Complete Sol. is $\boxed{y = \text{C.F} + \text{P.I.}}$

Que 3) Solve diff Eq. $\frac{d^2y}{dx^2} + y = 0$; given that $y(0) = 2$ and $y(\pi/2) = -2$.

Ans. Soln $y = 2(\cos x - \sin x)$

Que 4) $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y = 0$. under the Condition. $y(0) = 0$ $y'(0) = 0$ and $y''(0) = 2$.

Ans $\boxed{y = x^2 e^{-2x}}$

To find P.I. We have $f(D)y = X$

$$\therefore \boxed{P.I. = \frac{1}{f(D)} X}$$

General method of P.I.

If X is the function of x, then

$$\boxed{\frac{1}{(D-\alpha)} X = e^{\alpha x} \int X e^{-\alpha x}\, dx}$$

Remark ① $$\boxed{\frac{1}{(D+\alpha)} X = e^{-\alpha x} \int X e^{\alpha x}\, dx}$$

② If $\alpha = 0$ then from both cases we have

$$\boxed{\frac{1}{D} X = \int X\, dx}$$

③ $D \equiv \frac{d}{dx}$ = Differentiation with respect to x

$\frac{1}{D}$ = Integration with respect to x.

④ If $f(D) = (D-\alpha_1)(D-\alpha_2) \cdots (D-\alpha_n)$.

Then $$\frac{1}{f(D)} X = \frac{1}{(D-\alpha_1)(D-\alpha_2) - \quad (D-\alpha_n)} X$$

$$= \left\{ \frac{A}{(D-\alpha_1)} + \frac{A_2}{(D-\alpha_2)} + \cdots + \frac{A_n}{(D-\alpha_n)} \right\} X$$

on breaking into partial fractions

$$= A_1 \frac{1}{(D-\alpha_1)} X + A_2 \frac{1}{(D-\alpha_2)} X + \quad + A_n \frac{1}{(D-\alpha_n)} X$$

$$= A_1 e^{\alpha_1 x} \int X e^{-\alpha_1 x} dx + A_2 e^{\alpha_2 x} \int X e^{-\alpha_2 x} dx + \quad + A_n e^{\alpha_n x} \int X e^{-\alpha_n x} dx$$

⑤ The above method will be useful in case finding P.I. of $\sec ax$, $\operatorname{cosec} ax$, $\tan ax$, $\cot ax$ and any other forms which are covered by Short Method.

Que 1] Solve $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

Sol: A.E. is $m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm\sqrt{-a^2} = \pm\sqrt{a^2}\sqrt{-1} = \pm ai = 0 \pm ai$

$m = 0 \pm ai = \alpha \pm i\beta \quad (\alpha = 0,\ \beta = a).$

$C.F = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$

$= e^{0x}\left[c_1 \cos ax + c_2 \sin ax\right] = c_1 \cos ax + c_2 \sin ax$

$P.I. = \frac{1}{f(D)} X$

$= \frac{1}{(D^2+a^2)} X$

$= \frac{1}{(D+ai)(D-ai)} X$

$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \sec ax$

$= \frac{1}{2ai}\left[\frac{1}{(D-ai)}\sec ax - \frac{1}{(D+ai)}\sec ax\right] \longrightarrow (1)$

Now $\frac{1}{(D-ai)}\sec ax = e^{aix}\int \sec ax \cdot e^{-aix}\, dx$

$= e^{aix}\int \sec ax\,(\cos ax - i \sin ax)\, dx$

$= e^{aix}\int (1 - i\tan ax)\, dx$

$= e^{aix}\left[\int 1\, dx - i\int \tan ax\, dx\right]$

$= e^{aix}\left[x - i\,\frac{\log \sec ax}{a}\right]$

$= e^{aix}\left[x + \frac{i}{a}\log\cos ax\right]$

Similarly $\frac{1}{D+ai}\sec ax = e^{-aix}\left[x - \frac{i}{a}\log\cos ax\right]$

By eq (1), we get

$P.I. = \frac{1}{2ai}\left[e^{aix}\left(x + \frac{i}{a}\log\cos ax\right) - e^{-aix}\left(x - \frac{i}{a}\log\cos ax\right)\right]$

$= \frac{1}{2ai}\left[(e^{aix} - e^{-aix})x + \frac{i}{a}(e^{aix} + e^{-aix})\log\cos ax\right]$

$= \frac{1}{2ai}\left[x(2i\sin ax) + \frac{i}{a}(2\cos ax)\log\cos ax\right]$

$= \frac{2i}{2ai}\left[(x \sin ax) + \frac{1}{a}\cos ax \log \cos ax\right]$

$= \frac{1}{a}\left[x \sin ax + \frac{1}{a}\cos ax . \log \cos ax\right]$

So general solution or complete solution is $y = C.F + P.I.$

$y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a}\left(x \sin ax + \frac{1}{a}\cos ax \log \cos ax\right)$. Ans

Ques/ Find the general solution of diff Eq $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$.

Sol. Given Eq. can be put as $(D^2 + 3D + 2)y = e^{e^x}$

A.E. is $D^2 + 3D + 2 = 0$ when $D = m$

$\Rightarrow m^2 + 3m + 2 = 0$

$\Rightarrow (m+2)(m+1) = 0$

$\Rightarrow m = -1, -2.$ (Roots are real & distinct)

$\therefore C.F = c_1 e^{-1x} + c_2 e^{-2x}$.

$P.I. = \frac{1}{(D^2 + 3D + 2)} e^{e^x}$

$= \frac{1}{(D+1)(D+2)} e^{e^x}$

$= \frac{1}{1}\left[\frac{1}{(D+1)} - \frac{1}{D+2}\right] e^{e^x}$

$= \frac{1}{(D+1)} e^{e^x} - \frac{1}{(D+2)} e^{e^x}$

$= I_1 - I_2$ ———— (1)

Now

$I_1 = \frac{1}{(D+1)} e^{e^x}$ $\left[\frac{1}{D+\alpha} X = e^{-\alpha x}\int X e^{\alpha x} dx\right]$

$= e^{-x}\int e^{e^x} e^{+x} dx$

$= e^{-x}\left[\int e^t dt\right]$ Let $e^x = t$, $e^x dx = dt$

$= e^{-x}\left[e^t\right]$

$= e^{-x}\left[e^{e^x}\right] = e^{-x} e^{e^x}$

$I_2 = \frac{1}{(D+2)} e^{e^x} = e^{-2x}\int e^{e^x} e^{2x} dx$

Let $e^x = t$

$e^x dx = dt$

$$= e^{-2x}\left[\int e^t \cdot t \cdot dt\right]$$

$$= e^{-2x}\left[t(e^t) - 1(e^t)\right]$$

$$= e^{-2x}\left[(t-1)e^t\right]$$

$$= e^{-2x}\left[(e^x - 1)e^{e^x}\right]$$

$$= (e^{-x} - e^{-2x})e^{e^x}$$

by eq ① we get

$$P.I. = e^{-x}e^{e^x} - (e^{-x} - e^{-2x})e^{e^x}$$

$$= (\cancel{e^{-x}} - \cancel{e^{-x}} + e^{-2x})e^{e^x}$$

$$= e^{-2x}e^{e^x}$$

Hence complete solution is $y = C.F. + P.I.$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}e^{e^x}}$$

Que 3) Solve the diff Eq $\frac{d^2y}{dx^2} + y = x - \cot x$.

Ans $y = c_1 \cos x + c_2 \sin x + x - \sin x \log(\operatorname{cosec} x - \cot x)$

Short Methods for finding P.I. For diff Eq $f(D)y = X$, P.I. is given by

$P.I = \frac{1}{f(D)}X$, where X may be any one of the form e^{ax}, Ve^{ax}, $\sin(ax+b)$ $\cos(ax+b)$ x^m and $V.x$

$= \frac{1}{f(D)} X$ →
- → e^{ax}
- → $\sin(ax+b)$ or $\cos(ax+b)$
- → x^m ; $m = +ve$ Integer
- → $V.e^{ax}$; V is any function of x
- → $V.x$; V is any function of x.

V.V. Imp

Remark: P.I. by shorter method is very much shorter than general Method

(i) When $X = e^{ax}$ $\quad P.I. = \frac{1}{f(D)} e^{ax}$

$$\frac{1}{f(a)} e^{ax} \quad ; \quad f(a) \neq 0.$$

If $f(a) = 0$, then $\quad P.I. = x \frac{1}{f'(a)} e^{ax} \quad ; \quad f'(a) \neq 0$

Again if $f'(a) = 0$, then $\quad P.I. = x^2 \frac{1}{f''(a)} e^{ax} \quad ; \quad f''(a) \neq 0.$

and so on.

Que 1) Find P.I. of $(4D^2 + 4D - 3)y = e^{2x}$.

Sol.

$$P.I. = \frac{1}{(4D^2+4D-3)} e^{2x} = \frac{1}{(4 \cdot 2^2 + 4 \cdot 2 - 3)} e^{2x} \quad ; \quad \boxed{D = a = 2}$$

$$= \frac{1}{21} e^{2x}.$$

Que 2) Find P.I. of $(D^3 - 3D^2 + 4)y = e^{2x}$

Sol.

$$P.I. = \frac{1}{(D^3 - 3D^2 + 4)} e^{2x}$$

Put $D = 2$ in

$D^3 - 3D^2 + 4 = 8 - 3 \times 4 + 4$
$= 0$

so Rules is fail.

$$= x \frac{1}{(3D^2 - 6D + 0)} e^{2x}$$

Again put $D = 2$

in $3D^2 - 6D = 3 \times 4 - 6 \times 2 = 0$

Again fail

$$= x \cdot x \frac{1}{(6D - 6)} e^{2x}$$

Put $D = 2$, in

$6D - 6 = 6 \times 2 - 6 = 6$

$$= x^2 \frac{1}{6} e^{2x}$$

Que 2) Solve the diff. Eq. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2$

Sol:- A.E. is $m^3 - 3m^2 + 3m - 1 = 0$

$$(m-1)^3 = 0$$

$$\Rightarrow \quad m = 1, 1, 1$$

$$C.F. = (C_1 + xC_2 + x^2 C_3) e^x$$

$$P.I. = \frac{1}{(D-1)^3} (e^x + 2)$$

$$P.I = \frac{1}{(D-1)^3} e^x + \frac{1}{(D-1)^3} 2e^{0x} = I_1 + I_2 \quad \text{--- (1)}$$

$$I_1 = x \frac{1}{3(D-1)^2} e^x$$ because Rule is fail

$$= x \cdot x \cdot \frac{1}{6(D-1)} e^x$$ Again Rules is fail

$$= x \cdot x \cdot x \frac{1}{6} e^x$$

$$= \frac{x^3}{6} e^x.$$

$$I_2 = \frac{1}{(D-1)^3} 2e^{0x} = 2 \frac{1}{(D-1)^3} e^{0x}$$ Put $D=0$ in $(D-1)^3 = (-1)^3 = -1$

$$= -2e^0 = -2$$

By (1) $$\boxed{P.I. = \frac{1}{6} x^3 e^x - 2}$$

Complete solution is $y = CF + P.I$

$$\boxed{y = (c_1 + xc_2 + x^2 c_3) e^x + \frac{1}{6} x^3 e^x - 2}$$

Que 4) Solve $(D^2+D+1)y = (1+e^x)^2$.

Ans $y = e^{-x/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$.

Que 5) Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol: Hint. $\begin{cases} \sinh x = \dfrac{e^x - e^{-x}}{2} \\ \cosh x = \dfrac{e^x + e^{-x}}{2} \end{cases}$

Ans $y = c_1 e^{-2x} + (c_2 + c_3 x) e^x + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$

Que 6) Solve $y'' + 4y' + 13y = 18e^{-2x}$; $y(0) = 0$, $y'(0) = 9$

Ans $y = e^{-2x}(-2\cos 3x + 3\sin 3x + 2)$.

(ii) when $X = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I. = \frac{1}{f(D)} X$$

$$= \frac{1}{f(D)} \{\sin(ax+b) \text{ or } \cos(ax+b)\}$$

$$= \frac{1}{\phi(D^2)} \{\sin(ax+b) \text{ or } \cos(ax+b)\}$$

Put $D^2 = -a^2$

$$= \frac{1}{\phi(-a^2)} \{\sin(ax+b) \text{ or } \cos(ax+b)\}; \quad \text{Provided } \phi(-a^2) \neq 0.$$

If $\phi(-a^2) = 0$. Then Rule is fails. In this case

$$P.I = x \cdot \frac{1}{\phi'(-a^2)} \{\sin(ax+b) \text{ or } \cos(ax+b)\} \quad ; \quad \phi'(-a^2) \neq 0.$$

V.V. Imp

Que 1) Find P.I. of $(D^2+a^2)y = \sin ax$

Sol:

$$P.I = \frac{1}{(D^2+a^2)} \sin ax \quad ; \quad D^2 = -a^2$$

$$= \frac{1}{-a^2+a^2} \sin ax \quad \text{Rule fails}$$

$$= x \cdot \frac{1}{2D} \sin ax$$

$$= \frac{x}{2} \int \sin ax \, dx$$

$$= \frac{x}{2} \cdot \left(\frac{-\cos ax}{a}\right) = -\frac{x}{2a} \cos ax.$$

Que 2) Find P.I. & C.F of $(D^3+1)y = \sin(2x+1)$

Sol:- A.E. is $m^3 + 1 = 0$

$$(m+1)(m^2 - m + 1) = 0 \qquad \left(\because a^3+b^3 = (a+b)(a^2+b^2-ab)\right)$$

$m = -1, \quad m^2 - m + 1 = 0$

$$m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times1\times1}}{2\times1} = \frac{+1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Roots are $m = -1, \ \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

$$C.F = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

$$P.I. = \frac{1}{(D^3+1)} \sin(2x+1)$$

$$= \frac{1}{(D^2 \cdot D + 1)} \sin(2x+1)$$

Put $D^2 = -2^2 = -4$.

$$= \frac{1}{(-4D+1)} \sin(2x+1)$$

$$= \frac{1}{(1-4D)} \sin(2x+1)$$

By Rationalization so multiplying N^r & D^r by $(1+4D)$.

$$= \frac{1\,(1+4D)}{(1-4D)(1+4D)} \sin(2x+1)$$

$$= \frac{(1+4D)}{1-16D^2} \sin(2x+1) \quad ; \quad \text{Put } D^2 = -2^2 = -4$$

$$= \frac{(1+4D)}{1-16(-4)} \sin(2x+1)$$

$$= \frac{(1+4D)}{65} \sin(2x+1)$$

$$= \frac{1}{65}\left[1\cdot\sin(2x+1) + 4\,D\cdot\sin(2x+1)\right]$$

$$= \frac{1}{65}\left[\sin(2x+1) + 4\cdot\cos(2x+1)\cdot 2\right]$$

$$= \frac{1}{65}\left[\sin(2x+1) + 8\cos(2x+1)\right].$$

Complete sol $y = CF + P.I.$

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + \frac{1}{65}\left[\sin(2x+1) + 8\cos(2x+1)\right]$$

Que 3) Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$

Ans: $y = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x)$.

Que 4) Solve $(D^2+4)y = \cos^2 x$.

Hint Write $\cos^2 x = \left(\frac{1+\cos 2x}{2}\right)$ ✓

Ans $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(1 + x\sin 2x)$

Ques 5| Solve. $(D^2-4D+1)y = \cos x \cos 2x + \sin^2 x$

Sol.. A.E. is $m^2-4m+1=0$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$C.F = e^{2x}\left[c_1 \cosh\sqrt{3}x + c_2 \sinh(\sqrt{3}x)\right]$$

$$P.I. = \frac{1}{(D^2-4D+1)}(\cos x \cos 2x + \sin^2 x)$$

$$= \frac{1}{(D^2-4D+1)} \frac{1}{2} 2\cos x \cos 2x + \frac{1}{(D^2-4D+1)} \sin^2 x$$

$$= \frac{1}{2}\frac{1}{(D^2-4D+1)}(\cos 3x + \cos x) + \frac{1}{(D^2-4D+1)}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\frac{1}{(D^2-4D+1)}\cos 3x + \frac{1}{2(D^2-4D+1)}\cos x + \frac{1}{2}\frac{1}{(D^2-4D+1)} - \frac{1}{2(D^2-4D+1)}\cos 2x$$

$$P.I. = P_1 + P_2 + P_3 + P_4. \quad \text{——— (1)}$$

$$P_1 = \frac{1}{2}\frac{1}{(D^2-4D+1)}\cos 3x$$

$$= \frac{1}{2}\frac{1}{(-9-4D+1)}\cos 3x$$

$$= \frac{1}{2}\frac{1}{(-4D-8)}\cos 3x$$

$$= -\frac{1}{8}\frac{1}{(D+2)}\cos 3x$$

$$= -\frac{1}{8}\frac{(D-2)}{(D+2)(D-2)}\cos 3x$$

$$= -\frac{1}{8}\frac{(D-2)}{(D^2-4)}\cos 3x$$

$$= -\frac{1}{8}\frac{(D-2)}{(-9-4)}\cos 3x$$

$$= \frac{1}{104}(-3\sin 3x - 2\cos 3x)$$

$$P_2 = \frac{1}{2(D^2-4D+1)}\cos x \qquad D^2 = -1^2 = -1$$

$$= \frac{1}{2(-1-4D+1)}\cos x$$

$$= -\frac{1}{8}\frac{1}{D}\cos x$$

$$= -\frac{1}{8}\int \cos x\, dx$$

$$= -\frac{1}{8}\sin x$$

$$P_3 = \frac{1}{2}\frac{1}{(D^2-4D+1)} \cdot 1$$

$$= \frac{1}{2}\frac{1}{(D^2-4D+1)} \cdot e^{0x} \qquad D=0$$

$$= \frac{1}{2}\frac{1}{1}e^0 = \frac{1}{2}.$$

$$P_3 = -\frac{1}{2(D^2-4D+1)}\cos 2x = \frac{1}{-2(-4-4D+1)}\cos 2x = \frac{1}{2(4D+3)}\cos 2x$$

$$= \frac{1}{2(4D+3)}\frac{(4D-3)}{(4D-3)}\cos 2x$$

$$= \frac{1}{2} \frac{(4D-3)}{(16D^2-9)} \cos 2x \qquad D^2 = -4^2 = -4.$$

$$= \frac{1}{2} \frac{(4D-3)}{16(-4)-9} \cos 2x$$

$$= \frac{1}{2} \frac{(4D-3)}{-73} \cos 2x = \frac{1}{-146}(-8\sin 2x - 3\cos 2x)$$

$$= \frac{1}{146}(8\sin 2x + 3.\cos 2x).$$

$\therefore$ By Eq (1). $P.I. = -\frac{1}{104}(3\sin 3x + 2\cos 3x) + \frac{1}{8}\sin x + \frac{1}{2} + \frac{1}{146}(8\sin 2x + 3\cos 2x).$

Complete solution $y = CF + P.I.$

$$\boxed{y = e^{2x}(c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x) - \frac{1}{104}(3\sin 3x + 2\cos 3x) - \frac{1}{8}\sin x + \frac{1}{2} + \frac{1}{146}(8\sin 2x + 3\cos 2x).}$$

Q(46) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ and find the value of y when $x = \pi/2$ being given that $y = 3$, $\frac{dy}{dx} = 0$ when $x = 0$.

Sol:- We have $(D^2 + 2D + 10)y = -37\sin 3x.$

A.E is $m^2 + 2m + 10 = 0 \Rightarrow m = -1 \pm 3i$, $C.F = e^{-x}(c_1\cos 3x + c_2\sin 3x)$

$P.I. = \frac{1}{(D^2+2D+10)}(-37\sin 3x) = 6\cos 3x - \sin 3x.$

Complete sol $y = CF + P.I$

$$y = e^{-x}(c_1\cos 3x + c_2\sin 3x) + 6\cos 3x - \sin 3x \quad \text{——} (1)$$

From $y(0) = 3 \Rightarrow \boxed{c_1 = -3}$

Again $\frac{dy}{dx} = e^{-x}(-3c_1\sin 3x + c_2 3\cos 3x) - e^{-x}(c_1\cos 3x + c_2\sin 3x) + 6(-3)\sin 3x - 3\cos 3x$

From $\frac{dy}{dx} = 0$ when $x = 0$.

$\Rightarrow \boxed{c_2 = 0}$

Put these values in eq (1), we get $y = (6 - 3e^{-x})\cos 3x - \sin 3x$

when $x = \frac{\pi}{2}$, we get $y = -\sin\frac{3\pi}{2} = -(-1) = 1$. Ans

Case-III when $X = x^m$; $m \in N$

$$P.I. = \frac{1}{f(D)} x^m$$; Take lowest degree term common with signs from $f(D)$

and using the Binomial expansions

$$(1+x)^{-1} = 1 - x + x^2 + x^3 + \text{———} \infty$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \text{———} \infty.$$

Ques) Find P.I. of $(D^2 + 5D + 4)y = (x^2 + 7x + 9)$.

Sol:-

$$P.I = \frac{1}{(D^2+5D+4)}(x^2+7x+9)$$

$$= \frac{1}{4\left[1 + \frac{1}{4}(D^2+5D)\right]}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1 + \frac{1}{4}(D^2+5D)\right]^{-1}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1 - \frac{1}{4}(D^2+5D) + \frac{1}{16}(D^2+5D)^2 + \text{---}\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1 - \frac{1}{4}(D^2+5D) + \frac{1}{16}(D^4 + 25D^2 + 10D^3)\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1 - \frac{1}{4}(D^2+5D) + \underset{\times}{\frac{D^4}{16}} + 25\frac{D^2}{16} + \underset{\times}{\frac{10}{16}D^3}\right](x^2+7x+9)$$

Because they having degree more than 2.

$$= \frac{1}{4}\left[1 - \frac{1}{4}D^2 - \frac{5}{4}D + \frac{25}{16}D^2\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1 - \frac{5}{4}D + \frac{21}{16}D^2\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[(x^2+7x+9) - \frac{5}{4}D(x^2+7x+9) + \frac{21}{16}D^2(x^2+7x+9)\right]$$

$$= \frac{1}{4}\left[(x^2+7x+9) - \frac{5}{4}(2x+7) + \frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right).$$

Que 2) Solve $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$.

Ans $y = (c_1 + x c_2) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Case – IV When $X = V \cdot e^{ax}$; V = any function of x

$$P.I. = \frac{1}{f(D)} V \cdot e^{ax}$$

$$= e^{ax} \frac{1}{f(D+a)} V.$$

Que 1) Obtain general solution of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = (x-1)e^{2x}$

Sol:- A.E. is $(m^2 + 4m - 12) = 0$

$(m-2)(m+6) = 0$

$\Rightarrow m = 2, -6$

$C.F = c_1 e^{2x} + c_2 e^{-6x}$

$$P.I. = \frac{1}{(D^2+4D-12)} (x-1)e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 + 4(D+2) - 12} (x-1)$$

$$= e^{2x} \frac{1}{D^2 + 4 + 4D + 4D + 8 - 12} (x-1)$$

$$= e^{2x} \frac{1}{(D^2 + 8D)} (x-1)$$

$$= e^{2x} \frac{1}{8D\left[1 + \frac{D}{8}\right]} (x-1)$$

$$= e^{2x} \frac{1}{8} \frac{1}{D} \left[1 + \frac{D}{8}\right]^{-1} (x-1)$$

$$= \frac{e^{2x}}{8} \frac{1}{D} \left[1 - \frac{D}{8}\right] (x-1)$$

$$= \frac{e^{2x}}{8} \frac{1}{D} \left[(x-1) - \frac{1}{8} D(x-1)\right]$$

$$= \frac{e^{2x}}{8} \frac{1}{D} \left[(x-1) - \frac{1}{8}\right] = \frac{e^{2x}}{8} \frac{1}{D} \left(x - \frac{9}{8}\right)$$

$$= \frac{e^{2x}}{8} \int \left(x - \frac{9}{8}\right) dx$$

$$= \frac{e^{2x}}{8} \left[\frac{x^2}{2} - \frac{9}{8} x \right]$$

complete solution is $y = C.F + P.I.$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-6x} + e^{2x}\left(\frac{x^2}{16} - \frac{9x}{64}\right).}$$ Ans

Que 2) Solve $(D^2 - 2D + 1)y = x e^x \cos x$

Ans $y = (c_1 + c_2 x) e^x + e^x(-x \cos x + 2 \sin x).$

Que 3) Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x.$

Sol:- A.E. is $m^2 - 4m + 4 = 0$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow (m-2)(m-2) = 0 \Rightarrow m = 2, 2$$

$$C.F. = (c_1 + x c_2) e^{2x}.$$

$$P.I. = \frac{1}{(D-2)^2} 8 x^2 e^{2x} \sin 2x$$

$$= 8 \frac{1}{(D-2)^2} e^{2x} (x^2 \sin 2x)$$

$$= 8 \cdot e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 8 e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= 8 e^{2x} \iint x^2 \sin 2x \, dx \, dx$$

$$= 8 e^{2x} \int \left[x^2\left(-\frac{\cos 2x}{2}\right) - (2x)\left(-\frac{\sin 2x}{4}\right) + 2\left(\frac{\cos 2x}{8}\right)\right] dx$$

$$= 8 e^{2x} \int \left[-\frac{1}{2} x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \right] dx$$

$$= 8 e^{2x} \left[-\frac{1}{2} \int x^2 \cos 2x \, dx + \frac{1}{2} \int x \sin 2x \, dx + \frac{1}{4} \int \cos 2x \, dx \right]$$

$$= 8 e^{2x} \left[-\frac{1}{2} \left\{ x^2 \left(\frac{\sin 2x}{2}\right) - (2x)\left(\frac{-\cos 2x}{4}\right) + 2\left(-\frac{\sin 2x}{8}\right) \right\} \right.$$

$$+\frac{1}{2}\left\{x\left(-\frac{\cos 2x}{2}\right)-1\left(-\frac{\sin 2x}{4}\right)\right\}+\frac{1}{4}\left(\frac{\sin 2x}{2}\right)\Bigg]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[\left(-\frac{1}{4}x^2+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)\sin 2x - \frac{1}{2}x\cos 2x\right]$$

$$= 8e^{2x}\left[\left(\frac{3}{8}-\frac{x^2}{4}\right)\sin 2x - \frac{1}{2}x\cos 2x\right]$$

$$= e^{2x}\left[(3-2x^2)\sin 2x - 4x\cos 2x\right].$$

Hence complete sol is

$$y = CF + P.I$$

$$y = (c_1+xc_2)e^{2x} + e^{2x}\left[(3-2x^2)\sin 2x - 4x\cos 2x\right].$$

Ans

Remark: To find $P.I. = \frac{1}{f(D)}xV$

$$= \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V$$

Que Solve $(D^2+2D+1)y = x\cos x$

Sol:- A.E. is $m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1,-1.$

$$\therefore CF = (c_1+xc_2)e^{-x}.$$

$$P.I. = \frac{1}{(D^2+2D+1)}x\cos x = \left[x-\frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}\cos x$$

$$= \left[x - \frac{2D+2}{(D^2+2D+1)}\right]\frac{1}{(D^2+2D+1)}\cos x$$

$$= \left[x - \frac{2\cancel{(D+1)}}{(D+1)^{\cancel{2}}}\right]\frac{1}{(D+1)^2}\cos x$$

$$= \left[x - \frac{2}{(D+1)}\right]\frac{1}{(D^2+2D+1)}\cos x$$

$$= x \frac{1}{(D^2+2D+1)} \cos x - \frac{2}{(D^2+2D+1)(D+1)} \cos x$$

$$= x \frac{1}{(-1^2+2D+1)} \cos x - \frac{2}{(-1^2+2D+1)(D+1)} \cos x$$

$$= \frac{x}{2} \int \cos x \, dx - \cancel{2} \frac{1}{\cancel{2} D (D+1)} \cos x$$

$$= \frac{x}{2} \sin x - \frac{1}{(D^2+D)} \cos x$$

$$= \frac{x}{2} \sin x - \frac{1}{(D-1)(D+1)} \frac{(D+1)}{} \cos x$$

$$= \frac{x}{2} \sin x - \frac{1 \, (D+1)}{(D^2-1)} \cos x$$

$$= \frac{x}{2} \sin x - \frac{(D+1)}{(-1^2-1)} \cos x$$

$$= \frac{x}{2} \sin x + \frac{1}{2} (D+1) \cos x$$

$$= \frac{x}{2} \sin x + \frac{1}{2} [-\sin x + \cos x]$$

$$= \left(\frac{x}{2} - \frac{1}{2}\right) \sin x + \frac{1}{2} \cos x .$$

Complete sol. $y = CF + P.I. = (c_1 + x c_2) e^{-x} + \frac{1}{2} (x-1) \sin x + \frac{1}{2} \cos x .$

Cauchy's Euler Equations OR Homogeneous linear diff. Eq. with variable coefficients OR Equation Reducible to linear diff. Eq with constant coefficients

An Eq. of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad \text{—(1)}$$

or $$\left(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} x D + a_n\right) y = X$$

Where $a_0, a_1 \ldots a_n$ are constants and X is either constant or function of x, is called homogeneous linear diff Eq or Cauchy-Euler Equation.

Working Rule for finding solution of above Eq

Take $x = e^z$ or $z = \log x$, then

$$x \frac{dy}{dx} = D_1 y \quad ; \quad \text{where } D_1 \equiv \frac{d}{dz}$$

Similarly $$x^2 \frac{d^2 y}{dx^2} = D_1(D_1 - 1) y$$

$$x^3 \frac{d^3 y}{dx^3} = D_1(D_1 - 1)(D_1 - 2) y \quad \text{and so on.}$$

Que 1) Solve $$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - \lambda^2 y = 0 \quad \text{—(1)}$$

Sol: Given Equation is Cauchy-Euler Equation so we put $x = e^z$ or $z = \log x$ and

$$x \frac{dy}{dx} = D_1 y \quad ; \quad \text{where } D_1 \equiv \frac{d}{dz}$$

$$x^2 \frac{d^2 y}{dx^2} = D_1(D_1 - 1) y,$$

So Equation (1) becomes

$$\left[D_1(D_1 - 1) + D_1 - \lambda^2\right] y = 0$$

$$\left[D_1^2 - \cancel{D_1} + \cancel{D_1} - \lambda^2\right] y = 0$$

$\Rightarrow \quad (D_1^2 - \lambda^2) \cdot y = 0$; which is L.D. Eq. with constant coeff.

so A.E. is $\quad m^2 - \lambda^2 = 0$

$\Rightarrow \quad (m-\lambda)(m+\lambda) = 0$

$\Rightarrow \quad \boxed{m = \lambda, -\lambda}$

$\therefore \quad C.F = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$

$P.I. = 0.$

Complete sol. is $\quad y = C.F + P.I.$

$$y = c_1 e^{\lambda z} + c_2 e^{-\lambda z} + 0$$

$$y = c_1 (e^z)^\lambda + c_2 (e^z)^{-\lambda}$$

$$\boxed{y = c_1 x^\lambda + c_2 x^{-\lambda}}$$ Ans

Que 2) Solve $\quad x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x \quad \text{——} (1)$

sol:- Given Eq (1) is Cauchy-Euler Equation so we put

$x = e^z$ or $z = \log x$ and

$x \frac{dy}{dx} = D_1 y$; where $D_1 \equiv \frac{d}{dz}$

$x^2 \frac{d^2y}{dx^2} = D_1(D_1 - 1) y$

Put in (1), We get $\quad [D_1(D_1-1) + 4D_1 + 2] y = e^{e^z}$

$\Rightarrow \quad [D_1^2 - D_1 + 4D_1 + 2] y = e^{e^z}$

$\Rightarrow \quad (D_1^2 + 3D_1 + 2) y = e^{e^z}$

A.E. is $\quad m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$\Rightarrow \quad m = -1, -2$

$C.F = c_1 e^{-z} + c_2 e^{-2z}$

$$P.I. = \frac{1}{(D_1+1)(D_1+2)} e^{e^z} = \left(\frac{1}{D_1+1} - \frac{1}{D_1+2}\right) e^{e^z}$$

$$= \frac{1}{(D_1+1)} e^{e^z} - \frac{1}{(D_1+2)} e^{e^z}$$

$$= I_1 + I_2 \quad \text{—— (2)}$$

$$I_1 = \frac{1}{(D_1+1)} e^{e^z} =$$

$$\boxed{\frac{1}{D-\alpha} X = e^{\alpha x} \int X e^{-\alpha x} dx}$$

$$= e^{-z} \int e^{e^z} \cdot e^z \, dz$$

let $e^z = t$

$e^z dz = dt$

$$= e^{-z} \int e^t \, dt$$

$$= e^{-z}(e^t) = e^{-z}(e^{e^z})$$

$$I_2 = \frac{1}{(D_1+2)} e^{e^z} = e^{-2z} \int e^{e^z} \cdot e^{2z} \, dz$$

let $e^z = t$

$e^z \cdot dz = dt$

$$= e^{-2z} \int e^t \cdot t \cdot dt$$

$$= e^{-2z} \left[e^t (t-1) \right]$$

$$= e^{-2z} \left[e^{e^z} (e^z - 1) \right]$$

$$= e^{e^z} (e^{-z} - e^{-2z})$$

$$\therefore P.I = e^{-z} e^{e^z} - e^{e^z} (e^{-z} - e^{-2z})$$

$$= e^{e^z} (\cancel{e^{-z}} - \cancel{e^{-z}} + e^{-2z}) = e^{-2z} \cdot e^{e^z}.$$

complete sol. is $y = CF + P.I$

$$y = c_1 e^{-z} + c_2 e^{-2z} + e^{-2z} e^{e^z}$$

$$= c_1 x^{-1} + c_2 x^{-2} + x^{-2} e^x$$

$$\boxed{y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}}$$

Ans

Que solve $(x^3 D^3 + 3x^2 D^2 + x D + 1) y = x + \log x.$

Ans $y = c_1 x^{-1} + x^{1/2}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{1}{2}x + \log x$

Simultaneous linear differential Equations

Differential Equations in which two or more dependent Variables are function of single independent variable, are called simultaneous diff. Equations.

Exp suppose x & y are two dependent variables and t be the independent variable. Then

$$\left.\begin{aligned} f_1(D)x + f_2(D)y &= f(t) \\ g_1(D)x + g_2(D)y &= g(t) \end{aligned}\right\} \quad \text{———} \ (1)$$

where $D \equiv \frac{d}{dt}$, are simultaneous diff Equations.

Ques1) Solve $\frac{dx}{dt} + \omega y = 0$ and $\frac{dy}{dt} - \omega x = 0$. Also show that the points (x, y) lies on a circle.

Sol:- Let $D \equiv \frac{d}{dt}$. Then given Equations can be put as

$$Dx + \omega y = 0 \quad \text{———} \ (1)$$
$$-\omega x + Dy = 0 \quad \text{———} \ (2)$$

For eliminating y from (1) and (2), Multiplying (1) by D and (2) by ω and subtracting, we get

$$D^2 x + \cancel{\omega Dy} = 0 \times D \quad \text{———}$$
$$-\omega^2 x + \cancel{\omega Dy} = 0 \times \omega$$

Subtract $\quad (D^2 + \omega^2)x = 0$

A.E. is $\quad m^2 + \omega^2 = 0$

$$m^2 = -\omega^2$$
$$m = \pm i\omega = 0 \pm i\omega$$
$$x = e^{0t}\left[c_1 \cos\omega t + c_2 \sin\omega t\right]$$
$$\boxed{x = c_1 \cos\omega t + c_2 \sin\omega t} \quad \text{———} \ (3)$$

By eq (1), $\quad \omega y = -Dx \Rightarrow y = -\frac{1}{\omega} Dx$

$$y = -\frac{1}{\omega}\left[c_1(-\sin\omega t)\,\omega + c_2\,\omega\cos\omega t\right]$$

$$\boxed{y = c_1\sin\omega t + c_2\cos\omega t} \quad \text{———— (4)}$$

Thus Equations (3) and (4) together give the required solution of given Simultaneous Equations.

2nd Part:- We get $x = c_1\cos\omega t + c_2\sin\omega t$

$$y = c_1\sin\omega t + c_2\cos\omega t$$

For eliminating t, squaring and adding both sides, we get

$$x^2+y^2 = (c_1\cos\omega t + c_2\sin\omega t)^2 + (c_1\sin\omega t - c_2\cos\omega t)^2$$

$$= c_1^2\cos^2\omega t + c_2^2\sin^2\omega t + \cancel{2c_1c_2\sin\omega t\cos\omega t} +$$

$$c_1^2\sin^2\omega t + c_2^2\cos^2\omega t - \cancel{2c_1c_2\sin\omega t\cos\omega t}$$

$$= c_1^2(\cos^2\omega t + \sin^2\omega t) + c_2^2(\sin^2\omega t + \cos^2\omega t)$$

$$x^2+y^2 = c_1^2 + c_2^2$$

$$x^2+y^2 = \left(\sqrt{c_1^2+c_2^2}\right)^2$$

$$\boxed{x^2+y^2 = a^2} \qquad \text{Take } a = \sqrt{c_1^2+c_2^2}$$

Which is the Equation of circle. Thus the point (x,y) lies on circle.

Que 2) Solve the following Simultaneous diff Equations:

$$\frac{dx}{dt} + 5x - 2y = t$$

$$\frac{dy}{dt} + 2x + y = 0$$

given that $x = 0 = y$ when $t = 0$.

Sol:- Let $\frac{d}{dt} \equiv D$, so given Equations can be put as

$$(D+5)x - 2y = t \quad \text{———— (1)}$$

$$2x + (D+1)y = 0 \quad \text{———— (2)}$$

For eliminating y, multiplying Eq ① by $(D+1)$ and ② by 2 and ~~subtracting~~ Adding, we get

$$(D+1)(D+5)x - \cancel{(D+1)2y} = (D+1)t$$

$$2\cdot 2x + \cancel{2(D+1)y} = 2\cdot 0$$

Adding $\left[(D+1)(D+5)+4\right]x = (D+1)t + 0$

$$(D^2+5D+D+5+4)x = (D+t) \qquad \left(\because D = \frac{d}{dt}\right)$$

$$(D^2+6D+9)x = (1+t) \quad \text{——— ③}$$

A.E. is $m^2+6m+9=0$

$$(m+3)^2 = 0$$

$$\Rightarrow (m+3)(m+3)=0$$

$$\Rightarrow \boxed{m=-3,-3} \quad (\text{Repeated \& real})$$

$$C.F = (c_1+tc_2)e^{-3t}$$

$$P.I. = \frac{1}{(D^2+6D+9)}(1+t)$$

$$= \frac{1}{9\left[1+\left(\frac{D^2+6D}{9}\right)\right]}(1+t)$$

$$= \frac{1}{9}\left[1+\left(\frac{D^2+6D}{9}\right)\right]^{-1}(1+t) \qquad \left(\because (1+x)^{-1} = 1-x+x^2-\dots\infty\right)$$

$$= \frac{1}{9}\left[1-\left(\frac{D^2+6D}{9}\right)\right](1+t)$$

$$= \frac{1}{9}\left[(1+t) - \frac{6}{9}D(1+t) - \cancel{\frac{D^2}{9}}\right]$$

$$= \frac{1}{9}\left[(1+t)-\frac{6}{9}(1)\right] = \frac{1}{9}\left(1+t-\frac{2}{3}\right) = \frac{1}{9}\left(t+\frac{1}{3}\right)$$

Solution of Eq ③ is

$$x = C.F + P.I$$

$$\boxed{x = (c_1+tc_2)e^{-3t} + \frac{1}{9}\left(t+\frac{1}{3}\right)} \quad \text{——— ④}$$

From Eq ① $2y = (D+5)x - t$

$$= Dx + 5x - t$$

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$$2y = D\left\{(c_1+tc_2)e^{-3t} + \frac{1}{9}(t+1/3)\right\} + 5(c_1+tc_2)e^{-3t} + \frac{5}{9}(t+1/3) - t$$

$$2y = (c_1+tc_2)(-3e^{-3t}) + e^{-3t}(c_2) + \frac{1}{9} + 5(c_1+tc_2)e^{-3t} + \frac{5}{9}(t+1/3) - t$$

$$2y = 2(c_1+tc_2)e^{-3t} + e^{-3t}c_2 - \frac{4}{9}t + \frac{8}{27}$$

$$\boxed{y = (c_1+tc_2)e^{-3t} + \frac{c_2}{2}e^{-3t} - \frac{2}{9}t + \frac{4}{27}} \quad \text{——— (5)}$$

Equations (5) and (6), together give the general solution.

Again $x = 0$ when $t = 0$ ie

$$x(0) = 0$$

$$\Rightarrow (c_1 + c_2 \times 0)e^{-0} + \frac{1}{9}(0+1/3) = 0 \quad (\text{From eq } (4))$$

$$\Rightarrow \boxed{c_1 = -\frac{1}{27}}$$

Again from $y = 0$ when $t = 0$ ie $y(0) = 0$

$$\Rightarrow (c_1+0)e^{-0} + \frac{c_2}{2}e^{-0} - \frac{2}{9}\times 0 + \frac{4}{27} = 0 \quad (\text{From eq } (5))$$

$$\Rightarrow c_1 + \frac{c_2}{2} = -\frac{4}{27}$$

$$\frac{c_2}{2} = -\frac{4}{27} + \frac{1}{27} = \frac{-3}{27} = -\frac{1}{9}$$

$$\Rightarrow \boxed{c_2 = -\frac{2}{9}}$$

Hence put these values in eq (4) and (5), we get required particular solutions

$$\boxed{x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{9}(t+1/3)}$$

and

$$\boxed{y = -\frac{2}{27}(2+3t)e^{-3t} - \frac{2}{9}t + \frac{4}{27}}$$ Ans

Que 3) Solve $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$ and

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t$$

Sol. Let $D = \frac{d}{dt}$. Then given equations can be put as

$$(D^2+3)x + Dy = 4e^{-t} \quad \text{--- (1)}$$

$$-4Dx + (D^2+3)y = \sin 2t \quad \text{--- (2)}$$

Now multiplying Eq (1) by (D^2+3) and (2) by D. Then subtracting we get

$$(D^2+3)^2 x + \cancel{D(D^2+3)y} = (D^2+3)e^{-t}$$

$$-4D^2 x + \cancel{(D^2+3)Dy} = D\sin 2t$$

$$\left[(D^2+3)^2 + 4D^2\right]x = (D^2+3)e^{-t} - D\sin 2t$$

$$\left[D^4+9+6D^2+4D^2\right]x = 4e^{-t} - 2\cos 2t$$

$$(D^4+10D^2+9)x = 4e^{-t} - 2\cos 2t \quad \text{--- (3)}$$

A.E is $m^4 + 10m^2 + 9 = 0$

$(m^2+1)(m^2+9) = 0$

$\Rightarrow m^2+1=0$ & $m^2=-9$

$\Rightarrow m = \pm i, \pm 3i$

$$C.F = e^{0t}[c_1\cos t + c_2\sin t] + e^{0t}[c_3\cos 3t + c_4\sin 3t]$$

$$C.F. = c_1\cos t + c_2\sin t + c_3\cos 3t + c_4\sin 3t$$

$$P.I. = \frac{1}{(D^4+10D^2+9)}(4e^{-t} - 2\cos 2t)$$

$$= \frac{1}{(D^4+10D^2+9)}4e^{-t} - \frac{1}{(D^4+10D^2+9)}(2\cos 2t)$$

$$= 4\frac{1}{(1+10+9)}e^{-t} - 2\frac{1}{-2^2\cdot -2^2 + 10(-2^2)+9}\cos 2t$$

$$= \frac{4}{20}e^{-t} + \frac{2}{15}\cos 2t = \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

complete sol of (3) is $x = CF + P.I$

$$x = (c_1\cos t + c_2\sin t) + (c_3\cos 3t + c_4\sin 3t) + \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t. \quad \text{--- (4)}$$

Again for eliminating x from equations (1) & (2), for it we taking (1) $4D$ + (2) (D^2+3), we get

$$\left[(D^2+3)^2+4D^2\right]y = -4e^{-t} - \sin 2t$$

$$(D^4+10D^2+9)\,y = -4e^{-t} - \sin 2t \quad \text{———} (5)$$

A E is $\quad m^4+10m^2+9=0 \Rightarrow \quad m = \pm i, \ \pm 3i$

$$C.F = c_5\cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t$$

$$P.I. = \frac{1}{(D^4+10D^2+9)}(-4e^{-t}) - \frac{1}{(D^4+10D^2+9)}\sin 2t$$

$$= -\frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t.$$

Solution of (5) is

$$\therefore y = C.F + P.I$$

$$= c_5\cos t + c_6 \sin t + c_7\cos 3t + c_8 \sin 3t - \frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t \quad \text{———} (6)$$

Equations (4) & (6), together give the solution.

Que 4) Solve $\quad \dfrac{d^2x}{dt^2} + y = \sin t \quad \& \quad \dfrac{d^2y}{dt^2} + x = \cos t$

Ans $\quad x = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t)$

$$y = -c_1 e^{t} - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t - \cos t).$$

Linear differential Equation of 2nd order with variable coefficients

An Equation of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{——— (1)}$$

where P, Q and R are functions of x only, is called Linear diff Equation of 2nd order with variable coefficients.

This Equation can be solved by following methods

(1) Reduction of order
(2) Normal form
(3) Change of Independent Variable
(4) Method of Variation of Parameters (V. Imp)

(1) Reduction of order (or To find the solution of $y'' + Py' + Qy = R$ when part of C.F. is known)

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{——— (1)}$$

Case-(i) When $y = e^{mx}$ is part of C.F. Then it must satisfies the L.H.S. of (1)

$$\Rightarrow \frac{d^2}{dx^2}(e^{mx}) + P\frac{d}{dx}(e^{mx}) + Q(e^{mx}) = 0$$

$$\Rightarrow m^2e^{mx} + Pme^{mx} + Qe^{mx} = 0$$

$$\Rightarrow (m^2 + mP + Q)\cdot e^{mx} = 0 \qquad (\because e^{mx} \neq 0)$$

$$\Rightarrow m^2 + mP + Q = 0$$

if $m = 1$, then $1 + P + Q = 0$ and $y = e^x$ is Part of C.F.

if $m = -1$, then $1 - P + Q = 0$, " $y = e^{-x}$ is Part of C.F.

if $m = 2$, then $4 + 2m + Q = 0$ " $y = e^{2x}$ " "

if $m = -2$ then $4 - 2m + Q = 0$ " $y = e^{-2x}$ " "

Case (ii) When $y = x^m$ is part of C.F. Then it must satisfies the L.H.S. of (1) $\Rightarrow \frac{d^2}{dx^2}(x^m) + P\frac{d}{dx}(x^m) + Qx^m = 0$

$$\Rightarrow m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$$

$$\Rightarrow \left[m(m-1)+mPx+Qx^2\right]x^{m-2}=0 \qquad (\because x^{m-2}\neq 0)$$

$$\Rightarrow m(m-1)+mPx+Qx^2=0.$$

If $m=1$, $\quad P+Qx=0$, Then $y=x$ is Part of C.F.

If $m=2$, $\quad 2+2Px+Qx^2=0$, Then $y=x^2$ is " " , , ,

Working Rule:-

Step-I We put the given Eq. into standard form

$$1.\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R, \quad \text{we get}$$

$$P=(\;), \qquad Q=(\;), \qquad R=(\;)$$

Step-II Find the part of C.F. — Part of C.f

1. $m^2+mP+Q=0$ — $u=e^{mx}$
2. $m(m-1)+mPx+Qx^2=0$, — $u=x^m$

Step III complete solution $y=uv$, put in step (I) and we get

$$v''+\left(P+\frac{2u'}{u}\right)v'=\frac{R}{u}$$

Step-IV Put $v'=t$, then it becomes $t'+\left(P+\frac{2u'}{u}\right)t=R/u$

which is linear in t, where $R\neq 0$. solve it usual.

If $R=0$, then variable t and x will be separable.

in both cases, we find t.

Step-V put $t=\frac{dv}{dt}$ or $dv=t\,dt$

Int $\quad v=\int t\,dt+c_2$

Hence, complete solution $\boxed{y=uv}$

Que1 Solve $\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)y=e^x\sin x \quad -(1)$

Sol:- Given Equation is in standard form. so Comparing with

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

$\therefore \quad P = -\cot x, \quad Q = -(1-\cot x), \quad R = e^x \sin x$

Now $1+P+Q = 1-\cot x - 1 + \cot x = 0.$

$\therefore$ $u = e^x$ is the part of C.F. of ① and let $y = uv$ be the complete solution of ①. Now.

$y = e^x v$

$$\frac{dy}{dx} = e^x \frac{dv}{dx} + v e^x$$

$$\frac{d^2y}{dx^2} = e^x \frac{d^2v}{dx^2} + e^x \frac{dv}{dx} + v e^x + e^x \frac{dv}{dx} = e^x \frac{d^2v}{dx^2} + 2e^x \frac{dv}{dx} + e^x v$$

Put these values in eq ①, we get

$$e^x \frac{d^2v}{dx^2} + 2e^x \frac{dv}{dx} + e^x v - \cot x \left(e^x \frac{dv}{dx} + v e^x \right) - (1-\cot x) v e^x = e^x \sin x$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x \qquad \text{②}$$

Let $\frac{dv}{dx} = t$, then ② becomes

$$\frac{dt}{dx} + (2-\cot x)\, t = \sin x \qquad \text{③}$$

Which is Linear Eq. in t.

$$\frac{dy}{dx} + Py = Q$$

$$I.F = e^{\int P dx}$$

solution is

$$y(I.F) = \int Q(I.F)\, dx + \text{const}$$

$$I.F = e^{\int (2-\cot x)\, dx}$$

$$= e^{\int 2dx - \int \cot x\, dx}$$

$$= e^{2x - \log \sin x}$$

$$= \left(\frac{e^{2x}}{\sin x} \right)$$

Solution of ② is

$$t \cdot (I.F) = \int Q\, (I.F)\, dx + C_1$$

$$t \left(\frac{e^{2x}}{\sin x} \right) = \int \cancel{\sin x} \cdot \frac{e^{2x}}{\cancel{\sin x}}\, dx + C_1$$

$$t \left(\frac{e^{2x}}{\sin x} \right) = \int e^{2x}\, dx + C_1$$

$$t \left(\frac{e^{2x}}{\sin x} \right) = \frac{e^{2x}}{2} + C_1$$

$$t = \frac{1}{2}\sin x + C_1 \frac{\sin 2x}{e^{2x}}$$

$$t = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

$$\frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

Int
$$\int dv = \int\left(\frac{1}{2}\sin x + c_1 e^{-2x}\sin x\right)dv$$

$$v = \frac{1}{2}\int \sin x\,dx + c_1\int e^{-2x}\sin x\,dx$$

$$v = -\frac{1}{2}\cos x + c_1\frac{e^{-2x}}{(-2)^2+1^2}(-2\sin x - \cos x) + c_2$$

$$v = -\frac{1}{2}\cos x - \frac{1}{5}c_1 e^{-2x}(2\sin x + \cos x) + c_2$$

Hence complete solution is $y = uv$

$$y = e^x\left[-\frac{1}{2}\cos x - \frac{1}{5}c_1 e^{-2x}(2\sin x + \cos x) + c_2\right]$$

$$\boxed{y = -\frac{1}{2}e^x\cos x - \frac{1}{5}c_1 e^{-x}(2\sin x + \cos x) + c_2 e^x}$$

Ans

Que 2) Solve $(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0$ of which $y = x$ is a solution.

Sol: Given Eq is

$$(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0$$

First, we convert it into standard form. Making coeff of $\frac{d^2y}{dx^2}$ be 1 so Divide whole Eq. by coeff of $\frac{d^2y}{dx^2}$. we get

$$\frac{d^2y}{dx^2} - \left(\frac{x\cos x}{x\sin x + \cos x}\right)\frac{dy}{dx} + \left(\frac{\cos x}{x\sin x + \cos x}\right)y = 0. \quad -(1)$$

Here part of C.F. $= x$ (given)

$$\boxed{u = x}$$

Let $y = uv = xv$ be the solution of (1).

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Put these values in eq (1), we get

$$\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - \left(\frac{x\cos x}{x\sin x + \cos x}\right)\left(v + x\frac{dv}{dx}\right) + \left(\frac{\cos x}{x\sin x + \cos x}\right)xv = 0$$

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{x^2\cos x}{x\sin x + \cos x}\frac{dv}{dx} = 0$$

$$\Rightarrow \quad x\frac{d^2v}{dx^2} + \left(2 - \frac{x^2\cos x}{x\sin x + \cos x}\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)\frac{dv}{dx} = 0 \quad \text{——— (2)}$$

let $\frac{dv}{dx} = t$, then (2) becomes

$$\Rightarrow \quad \frac{dt}{dx} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)t = 0$$

Separate the Variables

$$\frac{dt}{t} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)dx = 0$$

Int

$$\int\frac{dt}{t} + \int\left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)dx = 0$$

$$\Rightarrow \quad \log t + 2\log x - \log(x\sin x + \cos x) = \log c_1$$

$$\Rightarrow \quad \log\left(\frac{t\cdot x^2}{x\sin x + \cos x}\right) = \log c_1$$

$$\Rightarrow \quad t\cdot x^2 = c_1(x\sin x + \cos x)$$

$$\Rightarrow \quad t = c_1\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = c_1\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)$$

$$dv = c_1\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)dx$$

Int

$$v = c_1\left[\int\frac{\sin x}{x}dx + \int\frac{\cos x}{x^2}dx\right]$$

$$= c_1\left[\frac{1}{x}\int\sin x\,dx - \int\left(\frac{d}{dx}\left(\frac{1}{x}\right)\int\sin x\,dx\right)dx + \int\frac{\cos x}{x^2}dx\right]$$

$$= c_1\left[-\frac{1}{x}\cos x - \cancel{\int +\frac{1}{x^2}\cos x\,dx} + \cancel{\int\frac{\cos x}{x^2}dx}\right]$$

$$v = -\frac{c_1}{x}\cos x + c_2$$

Hence complete sol. is $y = uv = x\left(-\frac{c_1}{x}\cos x + c_2\right)$

$$\boxed{y = -c_1\cos x + c_2 x}$$ Ans

Que 3) Solve $y'' - 4xy' + (4x^2-2)y = 0$ given that $y = e^{x^2}$ is solution.

Ans $y = e^{x^2}(xc_1 + c_2)$.

(2) Find the complete solution of $y''+Py'+Qy=R$ when it is reduced to Normal form $\left(\frac{d^2v}{dx^2} + Iv = S\right.$, where $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$ and $\left. S = R/u \right)$ OR Removal of first derivatives

When CF can not be determined by previous Method, then we reduce the given diff Eq. in Normal form

$$\frac{d^2v}{dx^2} + Iv = S$$

Where $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$, $S = R/u$, $u = e^{-\frac{1}{2}\int P\,dx}$

Working Rule:-

Step-I Put the given Eq. into standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

So we get $P = (\ )$, $Q = (\ )$, $R = (\ )$

Step II. Let $y = uv$ be the complete solution where

$u = e^{-\frac{1}{2}\int P\,dx}$ and v is given by $\frac{d^2v}{dx^2} + Iv = S$, where

$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$ and $S = R/u$.

Note:- (1) P = Multiple of even number.

(2) If I = Constant then Normal Eq. will be Linear diff Eq. of Constant coefficients.

and if $I = \frac{\text{Constant}}{x^2}$, then it becomes Cauchy-Euler Equation

Que 1) Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$.

Sol:- Comparing the given Eq. with standard from (1)

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore$ $P = -4x$, $Q = 4x^2 - 1$, $R = -3e^{x^2}\sin 2x$

Let $y = uv$ be the complete solution of (1). where

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int -4x\,dx} = e^{\int 2x\,dx} = e^{\frac{2x^2}{2}} = e^{x^2}.$$

and v is given by $\frac{d^2v}{dx^2} + Iv = S$, ——— (2)

where $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$

$$= (4x^2 - 1) - \frac{1}{2}(-4x) - \frac{1}{4}(-4x)^2$$

$$= 4x^2 - 1 + \frac{4}{2} - 4x^2$$

$$= \cancel{4x^2} - 1 + 2 - \cancel{4x^2} = 1.$$

$$S = R/u = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$$

By eq (2), we get

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

$$(D^2 + 1)v = -3\sin 2x \quad \text{——— (3)}$$

A.E. is $m^2 + 1 = 0$

$\Rightarrow$ $m = 0 \pm i$ $(\alpha = 0,\ \beta = 1)$

$$C.F. = e^{0x}[c_1 \cos x + c_2 \sin x]$$

$$P.I. = \frac{1}{(D^2+1)} - 3\sin 2x$$

$$= -3\frac{1}{(D^2+1)}\sin 2x \qquad \left(\because \underline{D^2 = -2^2 = -4}\right)$$

$$= -3\frac{1}{(-4+1)}\sin 2x$$

$$= -3\frac{1}{-3}\sin 2x = \underline{\underline{\sin 2x}}$$

Solution of (3) is $v = C.F. + P.I.$

$$v = c_1\cos x + c_2 \sin x + \sin 2x$$

Hence, complete sol. of given diff Eq is $y = uv$

$$\boxed{y = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x)}$$

Ans

Que 2) Solve $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)y = 0$ —— (1)

Sol: Comparing with standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore$ $P = \frac{1}{x^{1/3}}$, $Q = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}$, $R = 0$.

Let $y = uv$ be the solution of (1), where

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int x^{-1/3}dx} = e^{-\frac{1}{2}\left(\frac{x^{-1/3+1}}{-1/3+1}\right)} = e^{-\frac{1}{2}\frac{x^{2/3}}{2/3}}$$
$$= e^{-\frac{3}{4}x^{2/3}}$$

and v is given by $\frac{d^2v}{dx^2} + Iv = S$ —— (2)

where
$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$
$$= \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{2}\left(-\frac{1}{3}x^{-4/3}\right) - \frac{1}{4}x^{-2/3}$$
$$= \cancel{\frac{1}{4}x^{-2/3}} - \cancel{\frac{1}{6}x^{-4/3}} - \frac{6}{x^2} + \cancel{\frac{1}{6}x^{-4/3}} - \cancel{\frac{1}{4}x^{-2/3}}$$
$$= -\frac{6}{x^2}.$$

$$S = \frac{R}{u} = 0.$$

By Eq (2), Normal form is $\frac{d^2v}{dx^2} - \frac{6}{x^2}v = 0$

$$\Rightarrow \quad x^2\frac{d^2v}{dx^2} - 6v = 0 \quad \text{—— (3)}$$

Which is Cauchy-Euler Equation so put $x = e^z$ or $z = \log x$

and $x\frac{dv}{dx} = D_1 v$; where $D_1 \equiv \frac{d}{dz}$.

$x^2\frac{d^2v}{dx^2} = D_1(D_1-1)v$, in (3)

$\therefore$ (3) becomes

$$[D_1(D_1-1) - 6]v = 0$$
$$(D_1^2 - D_1 - 6)v = 0 \quad \text{—— (4)}$$

A.E. is $m^2 - m - 6 = 0 \Rightarrow (m+2)(m-3) = 0$

$m = 3, -2$

$\Rightarrow$ $CF = c_1e^{3z} + c_2e^{-2z}$, P.I. $= 0$

So solution of ④ is $v = C.F + P.I$

$$v = c_1 e^{3z} + c_2 e^{-2z} + 0$$

$$v = c_1 x^3 + c_2 x^{-2}$$

Hence, complete solution of given eq ① is $y = uv$

$$\boxed{y = e^{-\frac{3}{4}x^{2}}\left(c_1 x^3 + c_2 x^{-2}\right)}$$

Ques 3) Solve $x\frac{d^2y}{dx^2} + 2x^2\frac{dy}{dx} + x(x^2 - 8)y = x^3 e^{-x^2/2}$

Ans $y = e^{-\frac{1}{2}x^2}\left[c_1 e^{3x} + c_2 e^{-3x} + \frac{1}{9}\left(x^2 + 2/9\right)\right]$. Ans

③ 3rd Method To find complete solution of $y'' + Py' + Qy = R$ By changing the independent variable

consider $y'' + Py' + Qy = R$ ——— ①

Let independent variable x changed to z where $z = f(x)$

$$\therefore\ y' = \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = y_1 \cdot z'$$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(y_1 z') = y_1 z'' + z'\cdot\frac{dy_1}{dx}$$

$$= y_1 z'' + z'\frac{dy_1}{dz}\cdot\frac{dz}{dx}$$

$$= y_1 z'' + z' y_2 \cdot z'$$

$$= (z')^2 y_2 + z'' y_1$$

Put these values in eq ①, we get

$$(z')^2 y_2 + z'' y_1 + P[y_1 z'] + Qy = R$$

$$(z')^2 y_2 + (z'' + Pz')y_1 + Qy = R$$

$$\Rightarrow\ y_2 + \left\{\frac{z'' + Pz'}{(z')^2}\right\} y_1 + \frac{Q}{(z')^2}y = \frac{R}{(z')^2}$$

$$\Rightarrow\ \boxed{y_2 + P_1 y_1 + Q_1 y = R_1}$$

Where $P_1 = \frac{z'' + Pz'}{(z')^2}$, $Q_1 = \frac{Q}{(z')^2}$, $R_1 = \frac{R}{(z')^2}$.

~~Ques~~ **Working Rule!** - (1) This Method is useful when square root of Q is possible.

(2) Choose z, st $\left(\frac{dz}{dx}\right)^2 = Q$ (Note carefully that, we omit -ve sign of Q. This is extremely Important to find real values)

Ques) By changing the independent variable, solve the diff Eq.

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$$

Sol: Given Eq is $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$ —— (1)

Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$\therefore P = -\frac{1}{x}, \quad Q = 4x^2, \quad R = x^4.$

Choose z, such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$$\Rightarrow \frac{dz}{dx} = 2x$$

$$\Rightarrow dz = 2x\,dx$$

Int $\int dz = \int 2x\,dx$

$$\Rightarrow z = 2\frac{x^2}{2} = x^2$$

$$\Rightarrow \boxed{z = x^2}$$ (omit constant always)

$$z' = 2x$$

$$z'' = 2.$$

Change independent variable x to z, then transform Eq. is

$$y_2 + P_1 y_1 + Q_1 y = R_1 \quad \text{—— (2)}$$

where $P_1 = \frac{Pz' + z''}{(z')^2} = \frac{-\frac{1}{x}\times 2x + 2}{(2x)^2} = 0.$

$$Q_1 = \frac{Q}{(z')^2} = \frac{4x^2}{(2x)^2} = 1.$$

$$R_1 = \frac{R}{(z')^2} = \frac{x^4}{(2x)^2} = \frac{1}{4}x^2 = \frac{1}{4}z$$ (In terms of z)

By eq (2), we have

$$y_2 + 0 \cdot y_1 + 1 \cdot y = \frac{1}{4} z$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = \frac{1}{4} z \quad \text{——— (3)}$$

A.E. is $m^2 + 1 = 0$

$m = 0 \pm i$

$CF = e^{0z}[c_1 \cos z + c_2 \sin z]$

$$P.I = \frac{1}{(D^2+1)} \frac{z}{4}$$

$$= \frac{1}{4} \frac{1}{1[1+D^2]} z$$

$$= \frac{1}{4}[1 + D^2]^{-1} z$$

$$= \frac{1}{4}[1 - D^2] z = \frac{1}{4} z - 0 = \frac{1}{4} z$$

Solution of (3) is $y = CF + P.I.$

$$y = c_1 \cos z + c_2 \sin z + \frac{1}{4} z$$

$$= c_1 \cos x^2 + c_2 \sin x^2 + \frac{1}{4} x^2. \quad \underline{\underline{\text{Ans}}}$$

Que 2) Solve by method of changing the independent variable

$$\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y \cos^3 x = 2\cos^5 x$$

Sol:- Given Eq is $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y \cos^3 x = 2\cos^5 x$

We first, convert it into standard form. So multiplying whole Equation by $\cos x$, so, we get

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - (2\cos^2 x) y = 2\cos^4 x \quad \text{—(1)}$$

Comparing with standard form

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

$\therefore P = \tan x$, $Q = -2\cos^2 x$, $R = 2\cos^4 x$

Now choose z, such that $\left(\frac{dz}{dx}\right)^2 = \cos^2 x$

$$\Rightarrow \quad \frac{dz}{dx} = \cos x$$
$$\Rightarrow \quad dz = \cos x \, dx$$

Iut $\quad z = \sin x$

$$\frac{dz}{dx} = \cos x, \qquad \frac{d^2z}{dx^2} = -\sin x$$

$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} = \frac{\tan x \cdot \cos x - \sin x}{\cos^2 x} = \frac{\sin x - \sin x}{\cos^2 x} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-2\cos^2 x}{\cos^2 x} = -2$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x = 2(1-\sin^2 x) = 2(1-z^2)$$

Transform Equation is $\quad \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + 0 \cdot \frac{dy}{dz} - 2y = 2(1-z^2)$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} - 2y = 2(1-z^2). \qquad \text{—— (2)}$$

A.E. is $\quad m^2 - 2 = 0$

$$\Rightarrow \quad m = 0 \pm \sqrt{2}$$

$$CF = e^{0z}\left[c_1 \cosh(\sqrt{2}z) + c_2 \sinh(\sqrt{2}z)\right]$$

$$P.I. = \frac{1}{-2+D^2} 2(1-z^2)$$

$$= \frac{1}{-\cancel{2}\left[1 - \frac{D^2}{2}\right]} \cancel{2}(1-z^2)$$

$$= -\left[1 - \frac{D^2}{2}\right]^{-1}(1-z^2)$$

$$= -\left[1 + \frac{D^2}{2}\right](1-z^2) = -\left[(1-z^2) + \frac{1}{2}D^2(1-z^2)\right]$$

$$= -\left[1 - z^2 + \frac{1}{2}(-2)\right]$$

$$= z^2$$

So Solution of (2) is $y = CF + P.I.$

$$y = c_1\cosh(\sqrt{2}z) + c_2\sinh(\sqrt{2}z) + z^2$$

Complete sol. of given Eq is

$$\boxed{y = c_1\cosh(\sqrt{2}\sin x) + c_2\sinh(\sqrt{2}\sin x) + \sin^2 x}$$

Que 3) Solve by changing the independent variable

$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$

Ans $y = c_1 e^{\cos x} + c_2 e^{2\cos x} + \frac{e^{-\cos x}}{6}$. Ans

V.V. Imp

Method 4 (Method of Variation of Parameters) This method is applicable when C.F. is known. This method is quite general and applies to equation of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{——— (1)}$$

, where P, Q, R are functions of x only.

It gives $\boxed{P.I. = -y_1\int\frac{y_2 R}{W}dx + y_2\int\frac{y_1 R}{W}dx}$ ——— (2)

where y_1 & y_2 are solutions of $y'' + Py' + Qy = 0$.

and W = Wronskian of y_1 & y_2

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Que 1) Solve by Method of Variation of Parameters:

$$\frac{d^2y}{dx^2} + y = \tan x$$

Sol:- Given Eq. is $\frac{d^2y}{dx^2} + y = \tan x$ ——— (1)

Comparing with standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$\therefore$ $P = 0$, $Q = 1$, $R = \tan x$

For C.F. $(D^2+1)y = 0$.

A.E. is $D^2+1 = 0 \Rightarrow m^2+1=0 \Rightarrow m = 0 \pm i$

$$C.F. = e^{0x}[c_1\cos x + c_2\sin x]$$

$$C.F. = c_1\cos x + c_2\sin x$$

For P.I. let $y_1 = \cos x$, $y_2 = \sin x$

W = Wronskian of y_1 & y_2 = $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = \cos^2 x + \sin^2 x = 1.$$

$$P.I. = -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx$$

$$= -\cos x \int \frac{\sin x \cdot \tan x}{1} dx + \sin x \int \frac{\cos x \cdot \tan x}{1} dx$$

$$= -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \cancel{\cos x} \frac{\sin x}{\cancel{\cos x}} dx$$

$$= -\cos x \int \frac{1-\cos^2 x}{\cos x} dx + \sin x\,(-\cos x)$$

$$= -\cos x \left[\int \frac{1}{\cos x} dx - \int \cos x\, dx \right] - \sin x \cos x$$

$$= -\cos x \left[\int \sec x\, dx - \sin x \right] - \sin x \cos x$$

$$= -\cos x \left[\log(\sec x + \tan x) - \sin x \right] - \sin x \cos x$$

$$= -\cos x \log(\sec x + \tan x) + \cancel{\cos x \sin x} - \cancel{\sin x \cos x}$$

$$= -\cos x \log(\sec x + \tan x).$$

complete solution is $y = C.F + P.I.$

$$\boxed{y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)}$$

Ans

Qu 2 Solve by Method of ~~sepa~~ Variation of Parameters

$$(D^2-1)y = 2(1-e^{-2x})^{-1/2}.$$

Sol:- Given Eq. can be put $\frac{d^2y}{dx^2} - y = 2(1-e^{-2x})^{-1/2}$

Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$\therefore P = 0, \quad Q = -1, \quad R = 2(1-e^{-2x})^{-1/2}.$

For C.F. :– Put $\frac{d^2y}{dx^2} - y = 0.$

A.E. is $m^2 - 1 = 0 \Rightarrow \boxed{m = \pm 1}$

$\boxed{C.F. = y = c_1 e^x + c_2 e^{-x}}$; $\quad y_1 = e^x, \quad y_2 = e^{-x}$

For P.I.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

$$P.I = -y_1\int \frac{y_2 R}{W}dx + y_2\int \frac{y_1 R}{W}dx$$

$$= -e^x\int \frac{e^{-x}\cdot(+2)(1-e^{-2x})^{-1/2}}{(-2)}dx + e^{-x}\int \frac{e^{x}\, 2\,(1-e^{-2x})^{-1/2}}{(-2)}dx$$

$$= e^x\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx + e^{-x}\int \frac{e^{x}}{\sqrt{1-e^{-2x}}}dx$$

$$= e^x\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx - e^{-x}\int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx$$

$$= e^x\left(-\sin^{-1}(e^{-x})\right) - e^{-x}\frac{(e^{2x}-1)^{1/2}}{1}$$

$$= -e^x\sin^{-1}(e^{-x}) - e^{-x}(e^{2x}-1)^{1/2}$$

$$I_1 = \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx \quad \text{let } e^{-x}=t$$

$$-e^{-x}dx = dt$$

$$= \int \frac{-dt}{\sqrt{1-t^2}}$$

$$= -\int \frac{1}{\sqrt{1-t^2}}dt$$

$$= -\sin^{-1}(t) = -\sin^{-1}(e^{-x})$$

$$I_2 = \int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx$$

$$e^{2x}=t$$

$$2e^{2x}dx = dt$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t-1}}$$

$$= \frac{1}{2}\int (t-1)^{-1/2}dt$$

$$= \frac{1}{2}\frac{(t-1)^{-1/2+1}}{(-1/2+1)} = \frac{1}{2}\frac{(t-1)^{1/2}}{\frac{1}{2}}$$

$$= \frac{(t-1)^{1/2}}{1},$$

$$= \frac{(e^{2x}-1)^{1/2}}{1}$$

Complete solution is

$$y = C.F + P.I$$

$$\boxed{y = c_1e^x + c_2e^{-x} - e^x\sin^{-1}(e^{-x}) - e^{-x}(e^{2x}-1)^{1/2}}$$

<u>Ans</u>

<u>Que 3</u> Use the Method of Variation of Parameters to solve

$$x^2y'' + xy' - y = x^2e^x$$

<u>Sol.</u> Given Eq is $x^2y'' + xy' - y = x^2e^x$ —— (1)

First, we convert it into standard form so multiplying whole Eq by x^{-2}, we get

$$\Rightarrow y'' + \frac{1}{x}y' - \frac{1}{x^2}y = e^x \quad \text{—— (2)}$$

Comparing with $y'' + Py' + Qy = R$

$$\therefore P = 1/x, \quad Q = -1/x^2, \quad R = e^x$$

<u>For C.F.</u> Put $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$

$$x^2y'' + xy' - y = 0 \quad \text{—— (3)}$$

This is Cauchy-Euler Equation so put $x = e^z$ or $z = \log x$

and $xD = D_1$

$x^2D^2 = D_1(D_1-1)$ where $D_1 = \frac{d}{dz}$.

Equation (3) becomes $[D_1(D_1-1) + D_1 - 1]y = 0$

$\Rightarrow [D_1^2 - \cancel{D_1} + \cancel{D_1} - 1]y = 0$

$\Rightarrow (D_1^2 - 1)y = 0$

A. E. is $m^2 - 1 = 0$, $\Rightarrow m = \pm 1$.

$C.F. = c_1 e^{z} + c_2 e^{-z} = c_1 x + c_2 x^{-1}$.

<u>For P.I.</u> Let $y_1 = x$ & $y_2 = x^{-1}$ and $\underline{R = e^x}$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -\frac{2}{x}$$

$$P.I. = -y_1 \int \frac{y_2 R}{W} dx + y_2 \int \frac{y_1 R}{W} dx$$

$$= -x \int \frac{x^{-1} \cdot e^x}{-2/x} dx + x^{-1} \int \frac{x e^x}{-2/x} dx$$

$$= +\frac{x}{2} \int e^x dx + x^{-1}\left(-\frac{1}{2}\right) \int x^2 e^x dx$$

$$= \frac{x}{2} e^x - \frac{x^{-1}}{2}\left[x^2(e^x) - (2x)(e^x) + 2(e^x)\right]$$

$$= \frac{x}{2} e^x - \frac{x^{-1}}{2} e^x (x^2 - 2x + 2)$$

$$= e^x \left[\cancel{\frac{x}{2}} - \cancel{\frac{x}{2}} + \frac{2}{2} - x^{-1}\right] = e^x (1 - 1/x)$$

$\therefore$ Complete solution is $y = CF + P.I$

$$\boxed{y = c_1 x + c_2 x^{-1} + e^x (1 - 1/x).}$$

<u>Que 4)</u> Solve by Method of variation of parameters the following diff Equation

(i) $\frac{d^2y}{dx^2} + a^2 y = \sec ax$; Ans $y = c_1 \cos ax + c_2 \sin ax + \frac{\cos ax \log \cos ax}{a^2} + \frac{x}{a} \sin ax$

(ii) $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$. Ans $y = \left[\log\left(\frac{1+e^x}{e^x}\right) - e^{-x} + c_1\right] e^x + \left[-\log(1+e^x) + c_2\right] e^{-x}$

(iii) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$, Ans $y = [\log(e^{-x}+1) + c_1] e^x + [\log(1+e^{-x}) - (1+e^{-x}) + c_2] e^{2x}$

(iv) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$, Ans $y = c_1 + c_2 e^{2x} - \frac{e^x}{2} \sin x$