Basics of Complex Numbers

(B1)

Complex Numbers :-

A number of the form $x+iy$; where x & $y \in \mathbb{R}$, is called a complex number. It is denoted by z. So thus we have

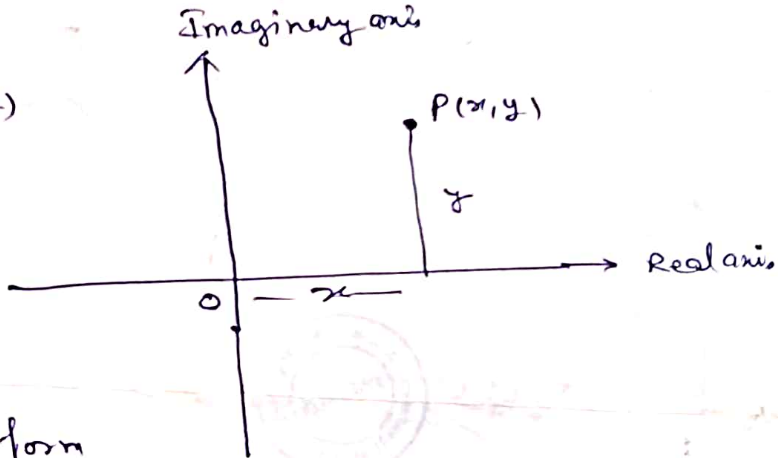
$$\boxed{z = a+ib} = \boxed{\underbrace{x}_{\text{Real part}}+i\underbrace{y}_{\text{Imaginary part}}} \; ; \text{ where } \boxed{i = \text{iota} = \sqrt{-1}}$$

Complex plane:

We know that every complex No. can be represented by a point in the complex plane or Argand plane or Gaussian plane

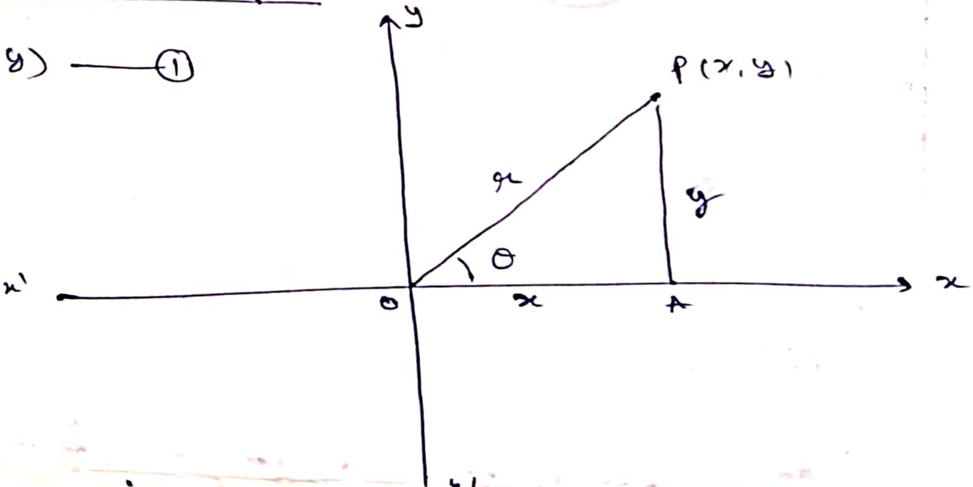
$$z = x+iy = (x,y)$$

Exp
$$z = -2+3i = (-2,3)$$
$$z = -i+1 = (-1,1)$$
$$z = i = (0,1),$$
$$z = 1 = 1+i\cdot 0 = (1,0)$$



Complex Number in polar form

$$z = x+iy = (x,y) \longrightarrow (1)$$



In Δ OPA

$$\cos\theta = \frac{x}{r}$$
$$\boxed{x = r\cos\theta}$$
& $$\sin\theta = \frac{y}{r}$$
$$\boxed{y = r\sin\theta}$$

$$z = r\cos\theta + ir\sin\theta$$
$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$
$$\boxed{z = re^{i\theta}}$$

Euler's theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Modulus & Amplitude of a Complex Number

let $z = x+iy \longrightarrow (1)$

where $x = r\cos\theta \longrightarrow (2)$

$y = r\sin\theta \longrightarrow (3)$

Squaring & Adding $x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

Dividing (3) by (2) we get

$$\tan\theta = y/x$$
$$\boxed{\theta = \tan^{-1}(y/x)}$$

Modulus of $z = |z| = \sqrt{x^2+y^2}$ (B2)

and Amplitude or Arguement of $(z) = \theta = \tan^{-1}(y/x)$.

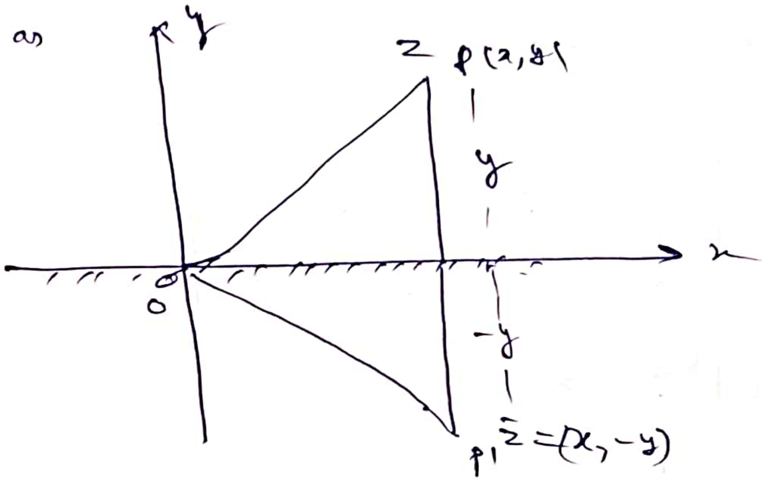
Amp (z) or Arg (z) $= \theta = \tan^{-1}(y/x)$

$$\boxed{z = \underbrace{r}_{|z|} e^{i\overbrace{\theta}^{\tan^{-1}(y/x)}}}$$

<u>Conjugate of a complex Numbers</u>

Let $z = x+iy$ be a complex No. Then it's congugate is denoted by $\bar{z}$ and define as

$$\boxed{\bar{z} = x - iy}$$



Again

$$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2+y^2$$

v.v.Imp

$$\boxed{z\bar{z} = |z|^2}$$

<u>Some other Important property</u>

(1)
$$z = x+iy$$
$$\bar{z} = x-iy$$

Add, we get $x = \left(\frac{z+\bar{z}}{2}\right) = Re(z)$, $\boxed{y = \frac{1}{2i}(z-\bar{z}) = Im(z)}$

(2) (a) $|z_1 z_2| = |z_1| |z_2|$. $\rightsquigarrow \{|z^n| = |z|^n\}$

(b) $Amp(z_1 z_2) = Amp(z_1) + Amp(z_2)$ — $\{Amp(z^n) = n\, Amp\, z\}$

(3) (a) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, (b) $Amp\left(\frac{z_1}{z_2}\right) = Amp(z_1) - Amp(z_2)$

(4) $|z_1+z_2| \leq |z_1| + |z_2|$ $\rightsquigarrow$ $|\Sigma| \leq \Sigma||$

(5) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(6) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

Unit-IV

Function of a complex Variables

Complex Variable :- The quantity $z = x+iy$, is called a complex variable, when x & y are two independent real variables.

Function of a Complex Variables :- Let $z = x+iy$ be the complex variable then $w = f(z)$ is called function of a complex variable.

For examples, $f(z) = z^2$ where $z = x+iy$ & $w = u+iv$ then

$$u+iv = (x+iy)^2$$
$$u+iv = x^2-y^2+i2xy$$
$$\therefore\ u = x^2-y^2 \quad \text{and} \quad v = 2xy$$

clearly u and v are functions of real variable x & y.

Thus $\boxed{w = f(z) = u(x,y) + i\,v(x,y)}$

If z expressed in polar form then u & v are functions of r and θ.

Function of a complex Variable

↓ Single Valued function | ↓ Multivalued function.

Single Valued function

Def:- If for every value of z, there corresponds a unique value of w, then w is called single valued function.

e.g. $w = z^2$ and $1/z = w$, are single valued function.

Multivalued function

Def:- If for every value of z there corresponds a more than one value of w, then w is called multivalued function.

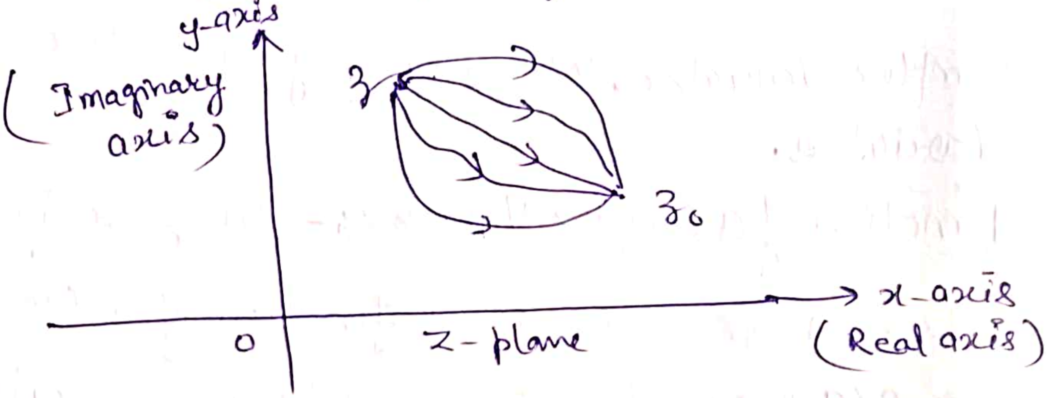
eg. 1. $w = z^{1/4}$ and $w = \text{Amp}(z)$, $(z \neq 0)$ are Multiple valued or Multivalued function of z.

$w = z^{1/4}$ is fourvalued &

$w = \text{Amp}(z)$ is infinite valued.

Continuity of $W = f(z)$:- Let $W = f(z)$ be a function of a complex variable z. Then $f(z)$ is called continuous at $z = z_0$ if (2)

$$\lim_{z \to z_0} f(z) = f(z_0) \text{ for each path of variation as } z \to z_0.$$



A function $f(z)$ is continuous in a region R of Z-plane then it is continuous at every point of the region R.

Differentiability of $W = f(z)$:- Let $W = f(z)$ be a function of a complex variable z. Then $f(z)$ is called differentiable at $z = z_0$ if

$$f'(z_0) = \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right] \text{ exists and } \quad \text{——— (1)}$$

Unique for each path of variation as $\Delta z \to 0$.

Let. $z_0 + \Delta z = z$ then (1) becomes

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)}} \quad \text{——— (2)}$$

This is the second form of derivative at $z = z_0$.

Again put $z_0 = 0$. Then

$$\boxed{f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}} \quad \text{——— (3)}$$

This is the formula for derivative at origin.

V. Imp

Exp 1 If $f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2} & ; z \neq 0 \\ 0 & ; z = 0. \end{cases}$ ③

Prove that $\underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z} \to 0$ (Along the radius Vector)

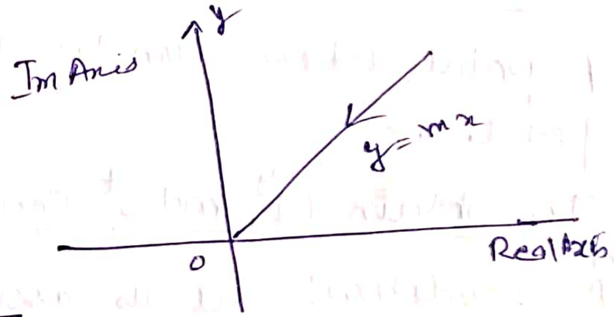
$\not\to 0$ (In any manner).

Sol:- Case-I: Let $z \to 0$ along the radius vector (or any straight line which passes through origin) $y = mx$.

Then $z = x+iy = x+imx = x(1+im)$, as $z\to 0$ then $x \to 0$.

Now

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z}$$

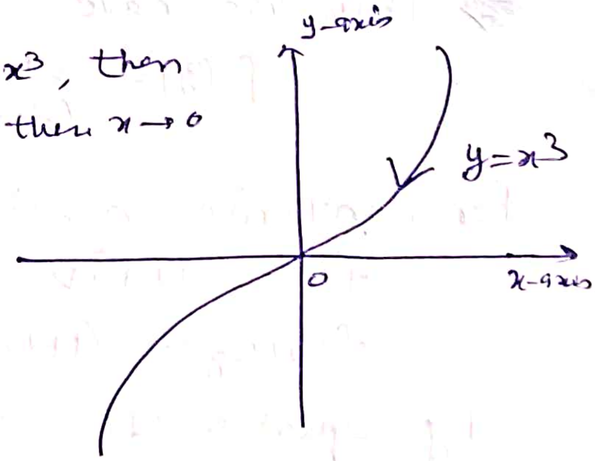


$$= \underset{z\to 0}{Lt} \frac{\dfrac{x^3 y(y-ix)}{x^6+y^2} - 0}{(x+iy)}$$

$$= \underset{x\to 0}{Lt} \frac{\dfrac{x^3\, mx(mx-ix)}{x^6+m^2x^2}}{x+imx} = \underset{x\to 0}{Lt} \frac{mx^4 \cancel{(x+imx)}(-i)}{x^2(m^2+x^4)\cancel{(1+im)x}}$$

$$= \underset{x\to 0}{Lt} \frac{mx^2(-i)}{(m^2+x^4)} = 0.$$

Case-II Let $z\to 0$ along the curve $y = x^3$, then $z = x+iy = x+ix^3$. when $z\to 0$ then $x\to 0$



$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z}$$

$$= \underset{z\to 0}{Lt} \frac{\dfrac{x^3 y(y-ix)}{x^6+y^2} - 0}{x+iy}$$

$$= \underset{x\to 0}{Lt} \frac{\dfrac{x^3 \cdot x^3(x^3-ix)}{x^6+x^6}}{(x+ix^3)} = \underset{x\to 0}{Lt} \frac{x^6(-i)\cancel{(x+ix^3)}}{2x^6\cancel{(x+ix^3)}} = -\frac{i}{2}$$

$\neq 0.$

We see that $f'(0)$ does not exist because $f'(0)$ is not unique for each path of variation as $z\to 0$. or $f(z)$ is not differentiable at $z=0$.

V.Imp **Analytic function**:- Let $w = f(z)$ be a function of a complex variable z. Then $w = f(z)$ is said to be analytic at $z = z_0$ if it is single valued & differentiable at the point $z = z_0$. OR (4)

A function $f(z)$ is said to be analytic at $z = z_0$ if it is differentiable at $z = z_0$ and at every point of some neighbourhood of z_0.

Analytic function is also known as holomorphic or regular function.

A point where function is not analytic is called a singular point.

To obtain N^y and S^t conditions for $f(z)$ to be an analytic

V.Imp **N^y Condition**:- Let us assume that $f(z)$ be an analytic in Region R

$\Rightarrow$ $f(z)$ is single valued & differentiable in R

$\Rightarrow$ $f'(z)$ exists and unique for each path of variation as $\Delta z \to 0$, in R

Thus we have

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right] \quad \text{——— (1)}$$

For convenience take

$$\left.\begin{aligned} f(z) &= u + iv \\ f(z+\Delta z) &= (u+\Delta u) + i(v+\Delta v) \end{aligned}\right\} \quad \text{——— (2)}$$

By equation (1) with (2), we get

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{u+\Delta u + i(v+\Delta v) - (u+iv)}{\Delta z} \right]$$

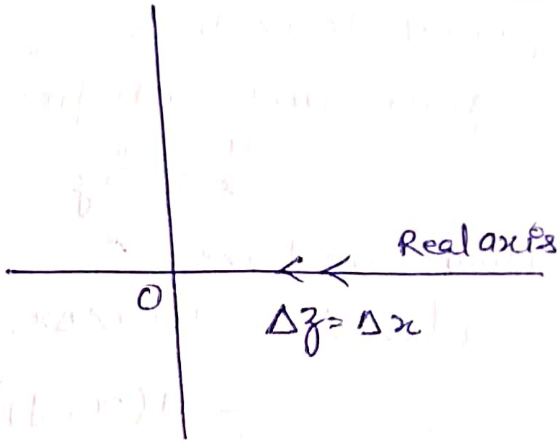
$$= \lim_{\Delta z \to 0} \left[\frac{\Delta u + i\Delta v}{\Delta z} \right] \quad \text{——— (3)}$$

Case-I Let $\Delta z \to 0$ along the real axis $\Delta y = 0$

$\therefore$

$$\Delta z = \Delta x + i\Delta y = \Delta x$$

By equation (5)



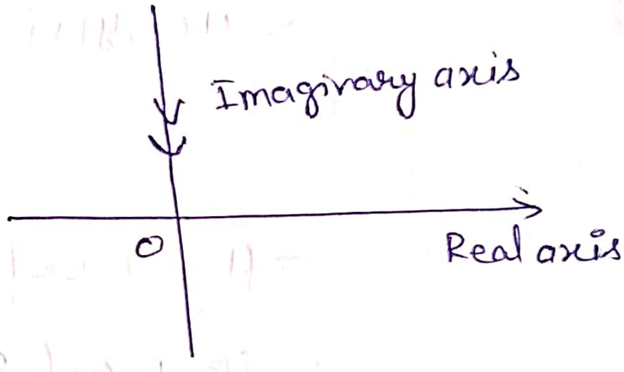
$$f'(z) = \underset{\Delta x \to 0}{Lt}\left[\frac{\Delta u + i\Delta v}{\Delta x}\right]$$

$$= \underset{\Delta x \to 0}{Lt}\frac{\Delta u}{\Delta x} + i\underset{\Delta x \to 0}{Lt}\frac{\Delta v}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{———— (4)}$$

Case-II Let $\Delta z \to 0$ along the Imaginary axis $\Delta x = 0$.

$$\therefore \Delta z = \Delta x + i\Delta y = i\Delta y$$



$$f'(z) = \underset{\Delta y \to 0}{Lt}\left[\frac{\Delta u + i\Delta v}{i\Delta y}\right]$$

$$= -i\underset{\Delta y \to 0}{Lt}\left(\frac{\Delta u}{\Delta y}\right) + \underset{\Delta y \to 0}{Lt}\left(\frac{\Delta v}{\Delta y}\right)$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{———— (5)}$$

By equation (4) and (5), Equating two values of $f'(z)$, we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts on both sides

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

or $\boxed{u_x = v_y \quad \& \quad u_y = -v_x}$

These above conditions are called Cauchy's Riemann Conditions or Equations.

Hence N^b Condition for $f(z)$ to be an analytic is that the C-R. Equations must be satisfied.

<u>St Conditions</u>:- Let $f(z)$ be a single valued function having partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of region R and satisfies C-R Conditions i.e

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Now, we have

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$= u(x,y) + \left(\Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y}\right) + \cdots$$

$$+ i\left[v(x,y) + \left(\Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y}\right) + \cdots\right]$$

Neglecting higher order terms of $\Delta x, \Delta y$ -

$$= u(x,y) + i\,v(x,y) + \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) + \cdots$$

$$= f(z) + \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)$$

$$= f(z) + \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\Delta y\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$= f(z) + (\Delta x + i\Delta y)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$= f(z) + \Delta z\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$f(z+\Delta z) - f(z) = \Delta z\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$\lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] = \lim_{\Delta z \to 0}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial}{\partial x}(u+iv) = \frac{\partial w}{\partial x}$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ are exists

Hence $f(z)$ is analytic function.

Cauchy's Riemann Conditions in polar form

We know that

$$W = f(z)$$

$$u + iv = f(re^{i\theta}) \quad \text{——— (1)}$$

P.D. w.r.t r eq (1), we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \quad \text{——— (2)}$$

P.D. w.r.t θ eq (1), we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) . re^{i\theta} i \quad \text{——— (3)}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad \text{By eq (2)}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Comparing real & imaginary part on both sides, we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

which are polar form of C-R- Conditions.

Harmonic function:- Let $H = H(x, y)$, then H is called Harmonic function if H satisfies Laplace Equation i.e. $\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H = 0$$

$$\Rightarrow \boxed{\nabla^2 H = 0}.$$

Thm If $f(z) = u + iv$ be an analytic function then prove that u and v both are harmonic function.

Proof:- Since $f(z) = u + iv$, is an analytic function, $\therefore$ C-R-Conditions are satisfied i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{——— (1)}$$

$$\& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{——— (2)}$$

Now $\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2)$ gives

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\Rightarrow$ u satisfies Laplace Equation

$\Rightarrow$ u is harmonic function

Again Equations (1) and (2) Rewrite as

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{——— (3)}$$

$$\& \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \text{——— (4)}$$

Now $\frac{\partial}{\partial x}(3) + \frac{\partial}{\partial y}(4)$ gives

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}$$

$$= 0$$

$\Rightarrow$ $\therefore$ v satisfies Laplace Equations

$\Rightarrow$ v is harmonic function.

Hence u and v both are harmonic function.

<u>Orthogonal curves</u>:- Two curves are said to be orthogonal to each other when they intersect at right angle at each point of their intersection.

<u>Thm</u>:- The analytic function $f(z) = u + iv$, consists two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$, which forms an orthogonal system of curves.

<u>Proof</u>:- Since $f(z)$ is an analytic function, $\therefore$ C-R-Conditions are satisfied i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\because f = f(x_1, x_2, \ldots x_n)$

$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$

Now from $u(x,y) = c_1$

$$du = 0$$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$m_1 = \frac{dy}{dx} = \frac{-\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \quad \text{——— (1)}$$

and from $v(x,y) = c_2$

$$dv = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$$m_2 = \frac{dy}{dx_2} = \frac{-\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} \text{ ——— } (2)$$

Now $m_1 \times m_2 = \frac{-\left(\frac{\partial u}{\partial x}\right)}{\cancel{\left(\frac{\partial u}{\partial y}\right)}} \times \frac{\cancel{-\left(\frac{\partial v}{\partial x}\right)}}{\cancel{\left(\frac{\partial v}{\partial y}\right)}} = -1$

$\therefore$ The given families of curves forms an orthogonal system of curves.

2007

Theorem:– An analytic function of constant modulus is constant.

Proof:– Since $f(z)$ is an analytic function, $\therefore$ C-R- conditions are satisfied i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Now, Here we are given $|f(z)| = \text{constant} = c$ (say)

$$|u+iv| = \text{constant} = c$$

$$\sqrt{u^2+v^2} = c \quad \text{squaring on both sides}$$

$$u^2+v^2 = c^2 \text{ ——— } (1)$$

P. D. w.r.to x, $2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \text{ ——— } (2)$$

Again P.D. w.r.to y, $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0 \text{ ——— } (3)$$

(By C-R- condition)

Squaring and adding equations (2) and (3), we get

$$\Rightarrow u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow (u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow c^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial y}$$

$\Rightarrow$ u and v both are constant.

$\Rightarrow$ u+iv is constant.

$\Rightarrow$ f(z) is constant. Proved

Que 1 | Given that $u(x,y) = x^2 - y^2$ and $v(x,y) = -\frac{y}{(x^2+y^2)}$ prove that both $u(x,y)$ & $v(x,y)$ both are harmonic functions but $u+iv$ is not analytic function of z.

Proof Given $u = x^2 - y^2$ & $v = -\frac{y}{(x^2+y^2)}$.

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

Now $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow$ u satisfies Laplace Equation, so u is harmonic function.

Again $$\frac{\partial v}{\partial x} = -y(-1)(x^2+y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2 \cdot 2y - 2xy\, 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)[x^2+y^2 - 4x^2]}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = -1\left[\frac{(x^2+y^2) \cdot 1 - y\, 2y}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)\, 2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{2y\cancel{(x^2+y^2)}[x^2+y^2 - 2y^2 + 2x^2]}{(x^2+y^2)^{\cancel{4}\,3}}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}.$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} + \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$ (14)

$$= \frac{2y}{(x^2+y^2)^3}\left[y^2-3x^2+3x^2-y^2\right]$$

$$=0.$$

$\Rightarrow$ v is harmonic

Hence u & v both are harmonic function.

3rd Part:- we see that

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C-R Conditions are not satisfies. $\therefore f(z)$ is not analytic.

Que 2) Show that the function $f(z) = u+iv = \sqrt{|xy|}$, is not analytic at origin, even though Cauchy's Reimann Conditions are satisfied at origin.

Sol:- Here given that

$$f(z) = \sqrt{|xy|}$$

$\boxed{z=0 \Leftrightarrow x=0=y}$

At origin:-

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\sqrt{|xy|}-f(0)}{x+iy}$$

$$= \lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy}$$

Let $z\to 0$ along the ~~real axis~~ line $y=mx$ then as $z\to 0$ becomes $x\to 0$.

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x\cdot mx|}}{x+imx}$$

$$= \lim_{x\to 0} \frac{\sqrt{|m|}}{(1+im)} = \frac{\sqrt{|m|}}{(1+im)} = \text{a function of } m.$$

We see that $f'(0)$, is dependend on m. $\therefore f'(0)$ does not have unique value for each path of variation as $z\to 0$. Hence $f'(0)$ does not exist.

$\therefore f(z)$ is not analytic at $z=0$.

At origin :- $u(x,y)=\sqrt{|xy|}$, $v(x,y)=0$

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{\sqrt{|x.0|}-0}{x}=\lim_{x\to 0}\frac{0-0}{x}=\lim_{x\to 0}0=0.$$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=\lim_{y\to 0}0=0$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=\lim_{x\to 0}0=0$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=\lim_{y\to 0}0=0.$$

We see that $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$\therefore$ C-R- conditions are satisfied at origin although ~~C-R conditions~~ ~~are not~~ $f(z)$ is not analytic at $z=0$.

Que 3) $$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}\ ;\ z\neq 0$$
$$=0\ ;\ z=0$$

Sol:- $$f(z)=\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}=u+iv$$

$$\therefore u(x,y)=\frac{x^3-y^3}{x^2+y^2},\quad v(x,y)=\frac{x^3+y^3}{x^2+y^2}$$

$u(x,y)$ & $v(x,y)$ both are rational function whose denominator is non-zero for every non-zero values of x & y. We know that ration function whose denominator is nonzero, is continuous.

$\therefore$ $u(x,y)$ & $v(x,y)$ are continuous.

$\Rightarrow$ $f(z)=u+iv$ is continuous.

At origin:-

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}$$

$$=\lim_{z\to 0}\frac{\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}-0}{x+iy}$$

Let $z\to 0$ along the line $y=mx$ then

$$f'(0)=\lim_{x\to 0}\frac{x^3(1+i)-m^3x^3(1-i)}{x^2(1+m^2)\,x(1+im)}$$

$$=\frac{(1+i)-m^3(1-i)}{(1+m^2)(1+im)}=f(m).$$

$f'(0)$ does not exists.

$\therefore f(z)$ is not analytic at $z=0$.

At origin:-

$$\frac{\partial u}{\partial x} = \underset{x\to 0}{Lt} \frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lt} \frac{x-0}{x} = \underset{x\to 0}{Lt}\ 1 = 1$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt} \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt} \frac{-y-0}{y} = \underset{y\to 0}{Lt}(-1) = -1.$$

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt} \frac{x-0}{x} = \underset{x\to 0}{Lt}(1) = 1$$

$$\frac{\partial v}{\partial y} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y} = \underset{x\to 0}{Lt} \frac{y-0}{y} = \underset{y\to}{Lt}\ 1 = 1.$$

We see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\therefore$ C-R- Conditions are satisfied. at origin.

At origin: C-R- Conditions are satisfied but $f(z)$ is not analytic at $z=0$.

Rules for Solving problems

(1) $f(z)$ is analytic function and then to show

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ continuous and

(iii) C-R- Conditions are satisfied $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

(2) If $f(z) = u+iv$ is analytic function and

(i) u is given then find v and $f(z)$.

(ii) v is given then find u and $f(z)$.

Que 4 | Show that $u(x,y) = x^3 - 4xy - 3xy^2$ is harmonic. Find it's harmonic conjugate $v(x,y)$ and the corresponding analytic function:-

Sol:-

$$u = x^3 - 4xy - 3xy^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 4y - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -4x - 6xy$$

$$\frac{\partial^2 u}{\partial y^2} = -6x$$

now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{6x} - \cancel{6x} = 0$

$\Rightarrow$ u is harmonic function.

2nd part:- Since function f(z) is analytic, $\therefore$ CR-conditions are satisfied i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ —(1) & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ —(2)

from $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$3x^2 - 4y - 3y^2 = \frac{\partial v}{\partial y}$$

Int $\partial v = \int (3x^2 - 4y - 3y^2)\partial y$

$$v = 3x^2y - 4\frac{y^2}{2} - 3\frac{y^3}{3} + f(x) \text{ (say)}$$

$$v = 3x^2y - 2y^2 - y^3 + f(x) \quad \text{—(3)}$$

Again from (2) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$-4x - 6yx = -[6xy + f'(x)]$$

$$f'(x) = -4x$$

$$\frac{d}{dx} f(x) = +4x$$

$$df(x) = +4x\,dx$$

Int $f(x) = +4\frac{x^2}{2} + c$

$$f(x) = +2x^2 + c$$

Put in (3), we get

$$\boxed{v = 3x^2y - 2y^2 - y^3 + 2x^2 + c}$$

3rd part:-

$$f(z) = u + iv$$

$$= x^3 - 4xy - 3xy^2 + i\,3x^2y - 2iy^2 - iy^3 + 2ix^2 + ic$$

$$= x^3 + (iy)^3 + 3xiy\,(x + iy) + 2i(x^2 - y^2 + 2ixy) + ic$$

$$= (x + iy)^3 + 2i\,(x + iy)^2 + ic$$

$$f(z) = z^3 + 2iz^2 + ic$$

(3) Milne's Thomson Method: If $f(z)$ is analytic and

(i) u is given then find $f(z)$ directly.

(ii) v is given then find $f(z)$ directly.

(iii) $u \pm v$ is given then find $f(z)$ directly.

Sol:- We have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{——— (1)}$$

(i) u is given then by C-R-Condition $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$\therefore f'(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) \quad \text{——— (2)}$$

Putting $y=0$ and $x=z$ in R.H.S of (1)

$$f'(z) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)$$

I. w.r to z

$$\boxed{f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0)\right]dz + c}$$

(ii) v is given then find $f(z)$

$$\boxed{f(z) = \int \left[\frac{\partial v}{\partial y}(z,0) + i\frac{\partial v}{\partial x}(z,0)\right]dz + c}$$

Que 5) If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ and $f(z) = ?$

Sol:

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) - \sin 2x(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2(\cosh 2y \cos 2x + 1)}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial x}(z,0) = \frac{2\cancel{(\cos 2z + 1)}}{(1+\cos 2z)^{\cancel{2}}} = \frac{2}{(1+\cos 2z)}$$

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial y}(z,0) = 0.$$

By Milne's Thomson Method

$$f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i \frac{\partial u}{\partial y}(z,0) \right] dz + c$$

$$= \int \frac{2}{1+\cos 2z} dz + c$$

$$= \int \frac{2}{\cancel{1}+2\cos^2 z - \cancel{1}} dz + c$$

$$= \int \sec^2 z + c = \tan z + c. \quad \underline{\text{Ans}}$$

Que 6] Determine an analytic function $f(z)$ in terms of z whose real part is $e^{x}(x \sin y - y \cos y)$. Ans $f(z) = i z e^{z} + c$.

Que 7] If $f(z) = u + iv$ is an analytic function, find $f(z)$ in terms of z if $u - v = e^{x}(\cos y - \sin y)$.

Sol:- Given $u - v = e^{x}(\cos y - \sin y)$ ——— (1)

P. D. w.r.to x $\quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^{x}(\cos y - \sin y)$ $\quad \left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right.$

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = e^{x}(\cos y - \sin y)$ ——— (2) $\quad \left. \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right)$

and partial differentiation w.r.to y eq (1)

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^{x}(-\sin y - \cos y)$$

$$\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = e^{x}(-\sin y - \cos y) \quad \text{——— (3)}$$

Adding (2) & (3), we get

$$2 \frac{\partial u}{\partial y} = e^{x}(\cancel{\cos y} - \sin y - \sin y - \cancel{\cos y})$$

$$\frac{\partial u}{\partial y} = -e^{x} \sin y$$

$$\frac{\partial u}{\partial y}(z,0) = -e^{z} \sin 0 = 0.$$

Subtract (3) from (2), we get

$$2 \frac{\partial u}{\partial x} = e^{x}(\cos y - \cancel{\sin y} + \cancel{\sin y} + \cos y)$$

$$\frac{\partial u}{\partial x} = e^{x} \cos y$$

$$\frac{\partial u}{\partial x}(z,0) = e^{z}.$$

By Milne's Thomson Method

$$f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i\,\frac{\partial u}{\partial y}(z,0)\right] dz + c$$

$$f(z) = \int e^{z}\, dz + c$$

$$\boxed{f(z) = e^{z} + c}$$

Que 8) If $f(z) = u + iv$ is an analytic function of z then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}.$$

Proof:- We have $z = x + iy$

$\bar{z} = x - iy$

Add $\quad x = \frac{1}{2}(z + \bar{z})$

Subtract $\quad y = \frac{1}{2i}(z - \bar{z}).$

If $f(z)$ is analytic function then

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f$$

$$\therefore \quad \frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ ——— } (1)$$

Similarly $\quad \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ ——— (2)

Multiplying (1) & (2), we get

$$\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\therefore \quad \boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}}$$

Que 9) Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f'(z)| = 0$

$$\text{L.H.S} = 4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\,\frac{1}{2}\log|f'(z)|^2 \qquad \because |z|^2 = z\cdot\bar{z}.$$

$$= 2\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\log\left\{f'(z)\cdot\overline{f'(z)}\right\}$$

$$= 2\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\log f'(z) + \log \overline{f'(z)}\right\}$$

$$= 2\,\frac{\partial}{\partial z}\left[0 + \frac{1}{\overline{f'(z)}}\cdot\overline{f''(z)}\right] = 0. = \text{R.H.S}$$

Que 10 Find the constants a, b, c, such that the function f(z) where $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$ is analytic. Express f(z) in terms of z. (18)

Sol: Given $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$

$$u = -x^2 + xy + y^2$$
$$v = ax^2 + bxy + cy^2$$

Since f(z) is analytic, ∴ C-R- Conditions are satisfies.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{——— (1)} \quad \&$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{——— (2)}$$

From (1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$-2x + y = bx + 2cy$$

Comparing the coeff of same ~~power~~ term on both sides, we get

$$\boxed{b = -2}, \quad 2c = 1 \Rightarrow \boxed{c = \tfrac{1}{2}}$$

and from (2) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$x + 2y = -[2ax + by]$$

$$x + 2y = -2ax - by$$

Comparing on both sides the coefficient of x & y, we get

$$-2a = 1 \Rightarrow \boxed{a = -\tfrac{1}{2}}$$

2nd part:- $f(z) = -x^2 + xy + y^2 + i\left(-\frac{1}{2}x^2 - 2xy + \frac{1}{2}y^2\right)$

$$= -x^2 + xy + y^2 - \frac{i}{2}(x^2 + 4xy - y^2)$$

$$= -x^2 + xy + y^2 - \frac{i}{2}x^2 - 2ixy + \frac{1}{2}iy^2$$

$$= -\left(1 + \frac{i}{2}\right)x^2 + \left(1 + \frac{i}{2}\right)y^2 + (1 - 2i)xy$$

$$= -\left(1 + \frac{i}{2}\right)\left[x^2 - y^2 - \frac{(1-2i)}{(2+i)}2xy\right]$$

$$= -\left(1 + \frac{i}{2}\right)\left[x^2 - y^2 + 2ixy\right]$$

$$= -\left(1 + \frac{i}{2}\right)\left[x + iy\right]^2$$

$$= -\left(1 + \frac{i}{2}\right)z^2 \quad \text{Ans}$$

Transformation or Mapping :- We choose two complex planes, call them z-plane and w-plane. In z-plane, we plot the point $z = x+iy$ and in w-plane, we plot the point $w = u+iv$. Thus the function $w = f(z)$ define a correspondence between the points of these two planes. Then the function $w = f(z)$ is a Mapping or transformation of z-plane into w-plane.

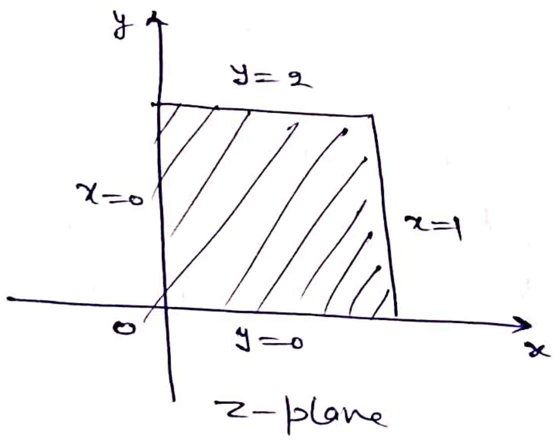
Example Consider the transformation $w = z+(1-i)$

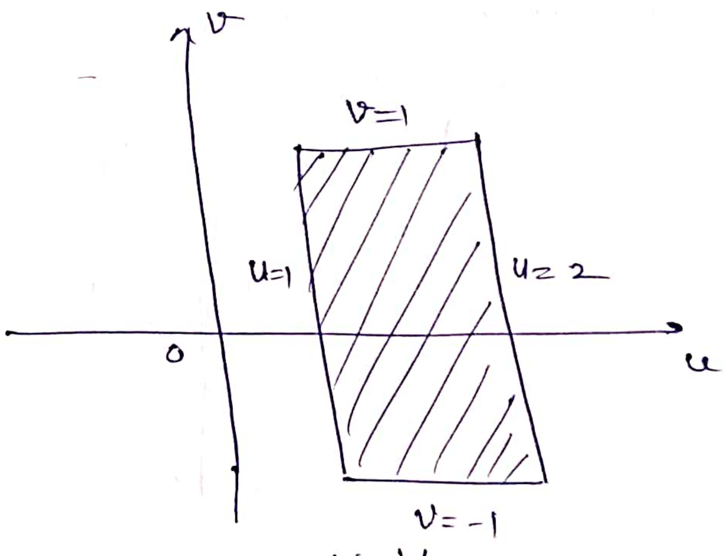
$$u+iv = (x+iy) + 1-i$$

$$u+iv = (x+1) + i(y-1)$$

$$\Rightarrow \quad u = x+1, \quad v = y-1,$$

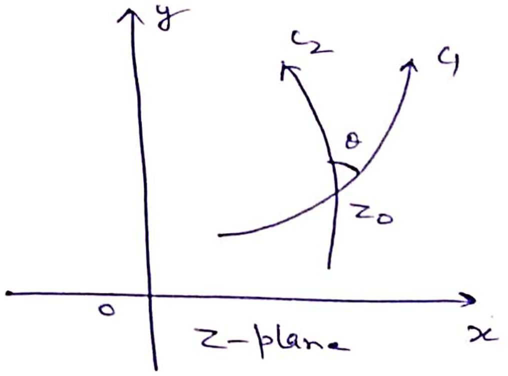
Then the lines $x=0$, $y=0$, $x=1$, $y=1$ in z-plane are Mapped onto the lines $u=1$, $v=-1$, $u=2$ and $v=1$ in w-plane.

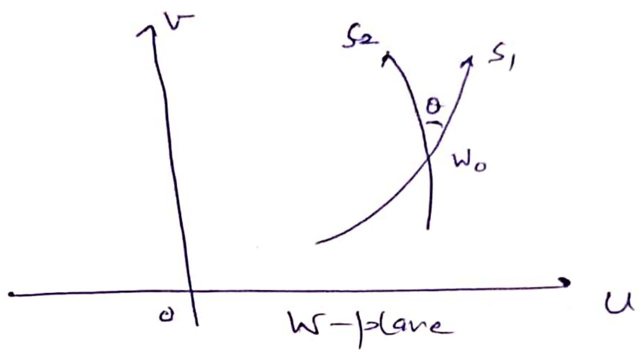




Conformal Mapping :- Let C_1 & C_2 be two curves in z-plane which intersect at z_0. Let $w = f(z)$ be the given transformation. Let S_1 & S_2 be the images of C_1 & C_2 in w-plane which intersects at w_0.

If the angle of Intersection between the images S_1 & S_2 is same as the angle of Intersection between C_1 & C_2 in both Magnitude & sense of ~~rotation~~ rotation. Then f is called Conformal Mapping

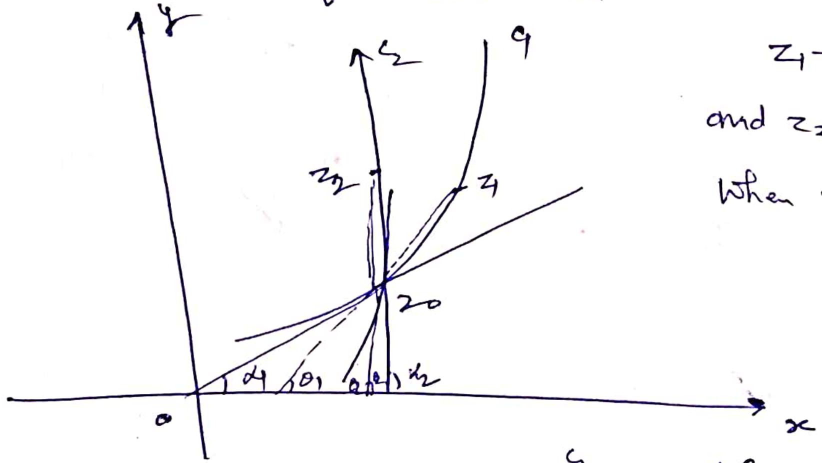




Isogonal Mapping :- A function that preserves the magnitude (size) of the angle but not sense is said to be isogonal.

Theorem :- If $w = f(z)$ be an analytic function and $f'(z) \neq 0$ in the region R of z-plane. Then $f(z)$ is conformal mapping.

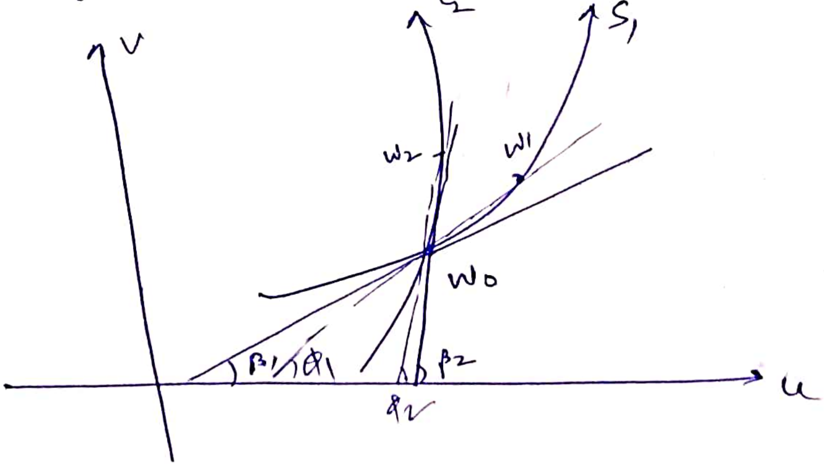
Proof :- Let $w = f(z)$ be an analytic function in the region R. Let c_1 & c_2 be two curves in z-plane, they are intersect at z_0. Draw the tangents at z_0, which makes an angle α_1 & α_2 with the x-axis. Let us take the points z_1 & z_2 on c_1 & c_2 at same distances r from z_0. Then



$$z_1 - z_0 = re^{i\theta_1} \quad \text{———} (1)$$

and $$z_2 - z_0 = re^{i\theta_2} \quad \text{———} (2)$$

When $r \to 0$ then $\theta_1 \to \alpha_1$ &
$\theta_2 \to \alpha_2$



In W-plane S_1 & S_2 be the images of C_1 & C_2 which intersect at point w_0 corresponding to z_0. Draw the tangents at w_0 which makes an angles β_1 & β_2 with u-axis. Let w_1 and w_2 be two points on S_1 & S_2 corresponding to z_1 & z_2. Then

$$w_1 - w_0 = \rho_1 e^{i\phi_1} \quad \text{———} (3)$$

and

$$w_2 - w_0 = \rho_2 e^{i\phi_2} \quad \text{———} (4)$$

When ρ_1 & $\rho_2 \longrightarrow 0$ then $\phi_1 \longrightarrow \beta_1$ & $\phi_2 \longrightarrow \beta_2$.

By definition of an analytic function

$$f'(z_0) = \underset{z_1 \to z_0}{Lt} \frac{f(z_1) - f(z_0)}{(z_1 - z_0)}$$

$$= \underset{z_1 \to z_0}{Lt} \frac{w_1 - w_0}{z_1 - z_0}$$

$$Re^{i\psi} = \underset{z_1 \to z_0}{Lt} \frac{\rho_1 e^{i\phi_1}}{r e^{i\theta_1}}$$

$$Re^{i\psi} = \underset{z_1 \to z_0}{Lt} \left(\frac{\rho_1}{r}\right) \cdot \underset{z_1 \to z_0}{Lt} e^{i(\phi_1 - \theta_1)}$$

$$\therefore \psi = Lt(\phi_1 - \theta_1) = Lt\,\phi_1 - Lt\,\theta_1 = \beta_1 - \alpha_1 \quad \text{———} (5)$$

Again $$f'(z_0) = \underset{z_2 \to z_0}{Lt} \frac{f(z_2) - f(z_0)}{(z_2 - z_0)}$$

$$= \underset{z_2 \to z_0}{Lt} \left(\frac{w_2 - w_0}{z_2 - z_0}\right)$$

$$Re^{i\psi} = \underset{z_2 \to z_0}{Lt} \frac{\rho_2 e^{i\phi_2}}{r e^{i\theta_2}}$$

$$= \underset{z_2 \to z_0}{Lt} \left(\frac{\rho_2}{r_2}\right) \cdot \underset{z_2 \to z_0}{Lt} e^{i(\phi_2 - \theta_2)}$$

$$\therefore \psi = Lt(\phi_2 - \theta_2) = Lt\,\phi_2 - Lt\,\theta_2 = \beta_2 - \alpha_2 \quad \text{———} (6)$$

By (1) and (6) $\Rightarrow$ $\beta_2 - \alpha_2 = \beta_1 - \alpha_1$

$\Rightarrow$ $\alpha_2 - \alpha_1 = \beta_2 - \beta_1$

$\Rightarrow$ f is Conformal mapping

Remark:- (1) A point at which $f'(z) = 0$ is called critical point of the transformation.

(2) A harmonic function remains harmonic under the Conformal mapping.

Coefficient of Magnification: – Coeff of magnification for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is given by $= |f'(\alpha + i\beta)|$.

Angle of rotation: – Angle of rotation for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is given by

$$= \text{Amp}\left[f'(\alpha + i\beta)\right].$$

Que 1) For the conformal mapping or transformation $w = z^2$, show that

(a) The coefficient of magnification at $z = 2+i$ is $2\sqrt{5}$.

(b) The angle of rotation at $z = 2+i$ is $\tan^{-1}(\frac{1}{2})$.

Sol:- We are given $w = f(z) = z^2$

$$f'(z) = 2z$$

$$f'(2+i) = 2(2+i) = 4+2i$$

(a) Coeff. of Magnification is $= |f'(2+i)| = |4+2i| = \sqrt{4^2+2^2} = \sqrt{20}$

$$= 2\sqrt{5}.$$

(b) Angle of rotation $= \text{Amp}\left[f'(2+i)\right]$

$$= \text{Amp}(4+2i)$$

$$= \tan^{-1}\left(\frac{2}{4}\right) = \tan^{-1}(0.5).$$

Que 2) If $u = 2x^2+y^2$ and $v = \frac{y^2}{x}$. Show that the curve u = constant, v = constant, cut orthogonally at all intersections But that the transformation $w = u+iv$ is not conformal.

Sol:- For the curve $u = \text{const} = k_1$ (say)

$$2x^2+y^2 = k_1$$

D.w.r.t x, $\quad 4x + 2y\frac{dy}{dx} = 0 \Rightarrow m_1 = \frac{dy}{dx} = -\frac{2x}{y} \quad \text{—(1)}$

for curve $v = \text{Const} = k_2$ (say)

$$\frac{y^2}{x} = k_2 \Rightarrow y^2 = k_2 x$$

D.w.r.t x $\quad 2y\frac{dy}{dx} = k_2$

$$m_2 = \frac{dy}{dx} = \frac{k_2}{2y} = \frac{y^2/x}{2y} = \frac{y}{2x}$$

$$m_1 \times m_2 = -\frac{2x}{y} \times \frac{y}{2x} = -1.$$

$\therefore$ curves cut orthogonally.

Again, since $\frac{\partial u}{\partial x} = 4x$, $\frac{\partial u}{\partial y} = 2y$

$\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}$, $\frac{\partial v}{\partial y} = \frac{2y}{x}$

$\because$ CR conditions are not satisfied, $\therefore$ $u+iv$ is not analytic so the transformation is not conformal.

Some standard transformation:-

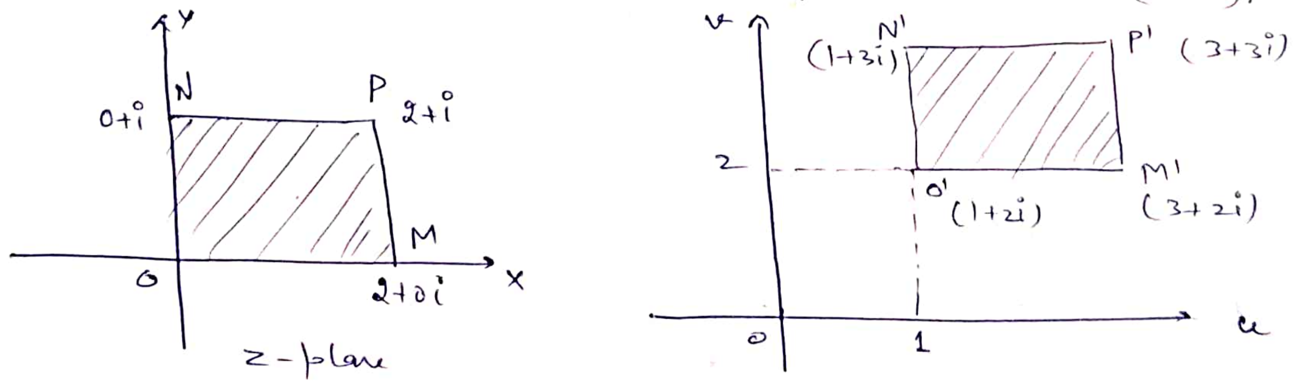
1. Translation:- Let $w = z + c$, where c is a complex constant —— (1)

Let $z = x+iy$, $c = a+ib$, $w = u+iv$, put in (1), we get

$$\Rightarrow u+iv = (x+iy) + (a+ib)$$
$$= (x+a) + i(y+b)$$
$$\Rightarrow u = x+a, \; \& \; v = y+b.$$

Thus the transformation is only translation of the axes and preserves the shape and size.

Example rectangle OMPN in z-plane is transformed to rectangle O'M'P'N' in w-plane under the transformation $w = z + (1+2i)$.



(2) Rotation:- $\boxed{w = ze^{i\theta_0}}$ figures in z-plane are rotated through an angle θ_0. If $\theta_0 > 0$, the rotation is anticlockwise and if $\theta_0 < 0$, the rotation is clockwise.

Example Consider the transformation $w = ze^{i\pi/4}$ and determine the region R' in w-plane corresponding to the triangular region R bounded by lines $x=0$, $y=0$ and $x+y=1$ in z-plane.

Sol: $W = z e^{i\pi/4}$

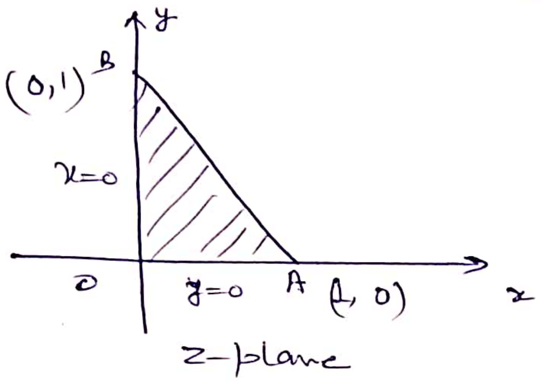
$$u + iv = (x+iy)\left(\cos \pi/4 + i \sin \pi/4\right)$$
$$= (x+iy)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}(x+iy)(1+i)$$
$$= \frac{1}{\sqrt{2}}\left[(x-y) + i(x+y)\right]$$

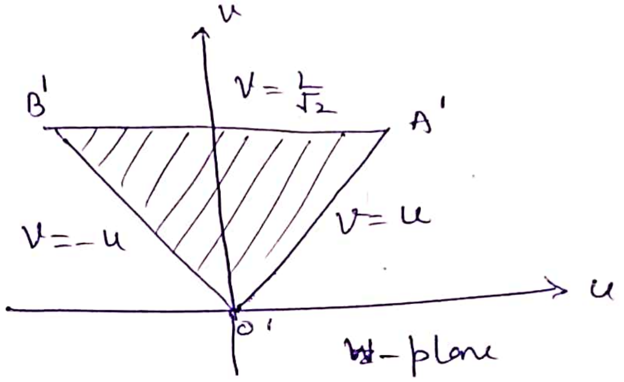
$\therefore \quad u = \frac{1}{\sqrt{2}}(x-y) \quad \& \quad v = \frac{1}{\sqrt{2}}(x+y)$

(i) Put $x=0$, $\quad u = -\frac{y}{\sqrt{2}}, \; v = \frac{y}{\sqrt{2}} \quad \Rightarrow \quad u = -v$ or $\boxed{v = -u}$

(ii) Put $y=0$, $\quad u = \frac{x}{\sqrt{2}}, \; v = \frac{x}{\sqrt{2}} \quad \Rightarrow \quad \boxed{v = u}$

(iii) $x+y=1$ then $\boxed{v = \frac{1}{\sqrt{2}}}$





③ **Magnification:-** $\boxed{W = cz}$ where c is real quantity.

(i) The figure in W-plane is magnified c-times the size of figure in z-plane

(ii) Both figures in z-plane and w-plane are similar.

Example Consider the transformation $W = 2z$ and determine the region R' of w-plane into which the triangular region R enclosed by lines $x=0$, $y=0$, $x+y=1$ in z-plane is mapped under mapping.

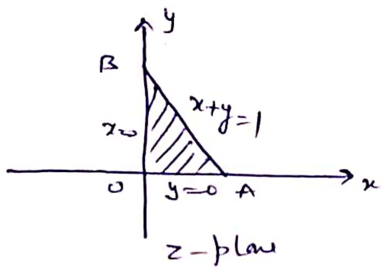
Sol:-

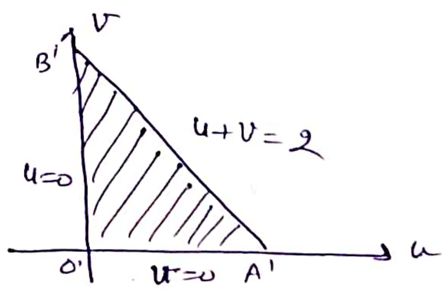
$$W = 2z$$
$$\Rightarrow u + iv = 2(x+iy)$$
$$\Rightarrow u + iv = 2x + i2y$$
$$\Rightarrow u = 2x \;\; \& \;\; v = 2y$$

$x = 0 \Rightarrow u = 0$

$y = 0 \Rightarrow v = 0$

$x+y=1 \Rightarrow u+v = 2$





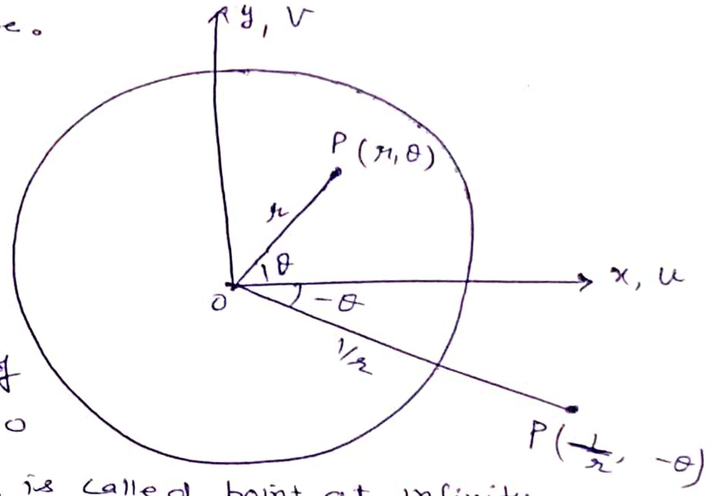
(4) Inverse :- $\boxed{W = \frac{1}{z}}$ ——— (1)

let $z = re^{i\theta}$ and $w = Re^{i\phi}$ put in (1), we get

$\therefore \quad Re^{i\phi} = \frac{1}{r} e^{-i\theta}$

$\therefore \boxed{R = \frac{1}{r}}$ and $\boxed{\phi = -\theta}$

The point $P(r, \theta)$ in z-plane is mapped onto the point $P'(\frac{1}{r}, -\theta)$ in w-plane.



Thus the transformation $W = \frac{1}{z}$ mapps the interior of unit circle $|z| = 1$ into exterior of the unit circle $|W| = 1$, and exterior of $|z| = 1$ into interior of $|W| = 1$. However the origin $z = 0$ is mapped to the point $W = \infty$, is called point at infinity.

Exp (1) Find the image of $|z - 3i| = 3$ under the mapping $w = \frac{1}{z}$.

Sol:- Given $|z - 3i| = 3$ —(1), $z = 1/w$ put in (1), we get

$\left|\frac{1}{w} - 3i\right| = 3 \implies |1 - 3iw| = 3|w|$

$\Rightarrow |1 - 3i(u + iv)| = 3|u + iv|$

$\Rightarrow |1 - 3iu - 3i^2 v| = 3|u + iv|$

$\Rightarrow |(1 + 3v) - 3iu| = 3|u + iv|$

$\Rightarrow \sqrt{(1+3v)^2 + (3u)^2} = 3\sqrt{u^2 + v^2}$

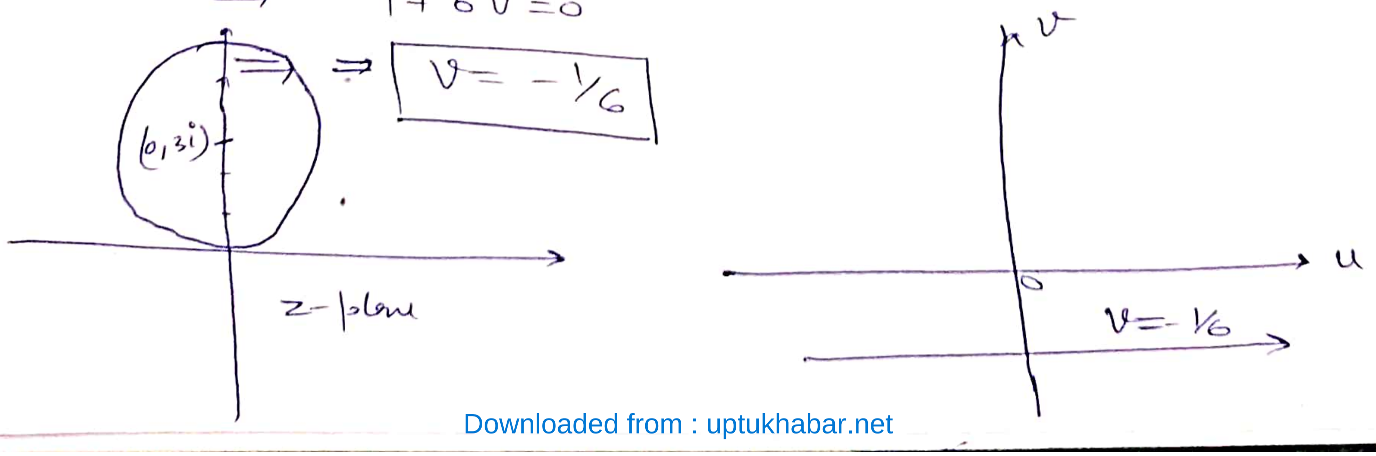
Square on both side

$\Rightarrow (1 + 3v)^2 + 9u^2 = 9(u^2 + v^2)$

$\Rightarrow 1 + \cancel{9v^2} + 6v + \cancel{9u^2} = \cancel{9u^2} + \cancel{9v^2}$

$\Rightarrow 1 + 6v = 0$

$\Rightarrow \boxed{v = -1/6}$



Exp 2 Find the image of Infinite strip $\frac{1}{4} \leq y \leq \frac{1}{2}$, under the transformation $w = 1/z$. Also show the regions graphically.

Sol: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x + iy = \frac{1}{(u+iv)} = \frac{(u-iv)}{(u+iv)(u-iv)}$

$$\Rightarrow x + iy = \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad \& \quad y = -\frac{v}{u^2+v^2}$$

$$y \leq \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} \leq \frac{1}{2} \Rightarrow -2v \leq u^2+v^2$$

$$\Rightarrow u^2+v^2+2v \geq 0$$

$$\Rightarrow u^2+v^2+2v+1 \geq 1$$

$$\Rightarrow u^2+(v+1)^2 \geq 1$$

Which represent outer portion of circle with Centre $(0,-1)$ & radius 1.

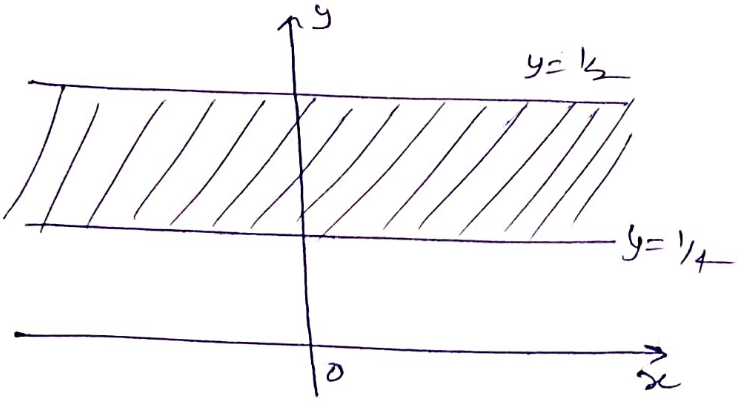
Also $\frac{1}{4} \leq y \Rightarrow \frac{1}{4} \leq \frac{-v}{u^2+v^2} \Rightarrow u^2+v^2 \leq -4v$

$$\Rightarrow u^2+v^2+4v \leq 0$$

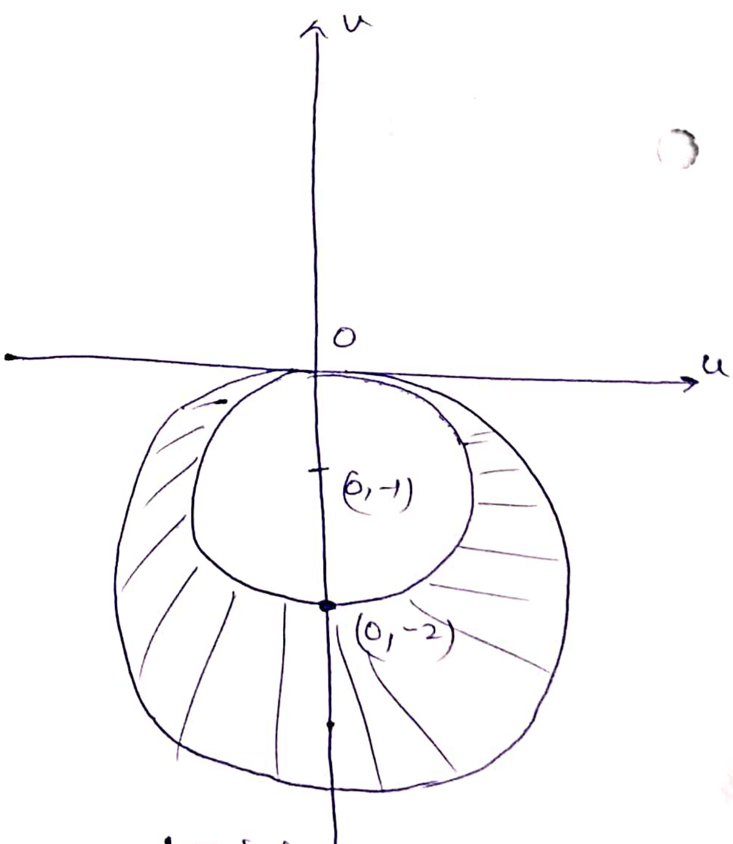
$$\Rightarrow u^2+v^2+4v+4 \leq 4$$

$$\Rightarrow u^2+(v+2)^2 \leq 2^2$$

Which represent the inner portion of the circle with Centre $(0,-2)$ and radius 2.



z-plane



w-plane

Bilinear transformation (or Möbius transformation or Linear fractional transformation)

A transformation of the form $\boxed{w = \dfrac{az+b}{cz+d}}$, where ——— (1)

a, b, c, d are complex constants such that $ad-bc \neq 0$, is called Bilinear (or Möbius or fractional) transformation.

Remark (1)

The transformation given by (1) is conformal, since

$$\frac{dw}{dz} = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

Remark (2)

The inverse mapping of (1) is

$$z = \left(\frac{-dw+b}{cw-a}\right);$$ which is also a ——— (2)

linear transformation.

Remark (3) Transformation (1) can be put as

$$cwz + wd - az - b = 0$$

which is linear in w and z and hence the Name bilinear transformation.

Remark (4) The expression $ad-bc$ is called determinant of Bilinear transformation.

Remark (5) $w = \dfrac{az+b}{cz+d} = \dfrac{az+b}{c(z+d/c)}$. ——— (1)

from (1) it is clear that each point in z-plane except the point $z = -d/c$ maps into unique point in w-plane.

Similarly from (2), each point in w-plane except $w = a/c$ maps into a unique point in z-plane.

Remark (6) Every Bilinear transformation $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ is the combination of Basic transformations

(1) Translation $\boxed{w = z+c}$

(2) Rotation $\boxed{w = z e^{i\theta_0}}$

(3) Magnification $\boxed{w = cz}$

(4) Inverse $\boxed{w = \frac{1}{z}}$

Fixed (or Invariant) points of Bilinear transformation:

We know that $w = \frac{az+b}{cz+d} = f(z)$ — (1)

If z-maps into itself, then $\boxed{w=z} \Rightarrow \boxed{f(z)=z}$

(1) becomes $\boxed{\frac{az+b}{cz+d} = z}$ — (2)

Roots of Eq (2) are called fixed points of Bilinear transformation.

If Roots are equal then bilinear transformation is said to be parabolic.

Cross-Ratio: — If four points z_1, z_2, z_3, z_4 are taken in order then the ratio

$$\frac{(z_1-z_2)}{(z_2-z_3)}\frac{(z_3-z_4)}{(z_4-z_1)} = \frac{(w_1-w_2)}{(w_2-w_3)}\frac{(w_3-w_4)}{(w_4-w_1)}$$

is called cross-Ratio. This is also put as in form

$$(z_1 z_2 z_3 z_4) = (w_1 w_2 w_3 w_4).$$

Proof: let $w = \frac{az+b}{cz+d}$ be the given transformation

$w_1 = \frac{az_1+b}{cz_1+d}$ $ad-bc \neq 0$

$w_2 = \frac{az_2+b}{cz_2+d}$

$w_3 = \frac{az_3+b}{cz_3+d}$

$w_4 = \frac{az_4+b}{cz_4+d}$

$w_1 - w_2 = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$, $(w_2-w_3) = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$

$w_3 - w_4 = \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}$, $(w_4-w_1) = \frac{(ad-bc)(z_4-z_1)}{(cz_4+d)(cz_1+d)}$

R.H.S = we get easily L.H.S.

Remark (1) A Bilinear transformation maps circles into circles

(2) A Bilinear transformation preserves cross ratio of four points

Ques 1) Find the Bilinear transformation which maps the points $z = 1, i, -1$ into the point $w = i, 0, -i$. Hence find image of $|z| < 1$.

Sol: Given $z_1 = 1, \quad z_2 = i, \quad z_3 = -1, \quad z_4 = z$

$w_1 = i, \quad w_2 = 0, \quad w_3 = -i, \quad w_4 = w.$

By cross Ratio $(z_1 z_2 z_3 z_4) = (w_1 w_2 w_3 w_4)$

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}$$

$$\frac{(1-i)(-1-z)}{(i+1)(z-1)} = \frac{(i-0)}{(0+i)}\frac{(-i-w)}{(w-i)}$$

$$-\frac{(1-i)(1+z)}{(i+1)(z-1)} = \frac{i}{i}\frac{-(i+w)}{(w-i)} \Rightarrow \frac{w+i}{w-i} = \frac{(1-i)}{(i+1)}\left(\frac{1+z}{z-1}\right)$$

$$\frac{z+1}{z-1} = -\frac{(i+1)}{(1-i)}\frac{(i+w)}{(w-i)}$$

$$= -\frac{(i+1)(1+i)}{(1-i)(1+i)}\frac{(i+w)}{(w-i)}$$

$$= -\frac{(i^2+1+2i)}{(1-i^2)}\left(\frac{w+i}{w-i}\right)$$

$$= -\frac{(-\cancel{1}+\cancel{1}+2i)}{2}\left(\frac{w+i}{w-i}\right)$$

$$= \frac{-(w+i)i}{(w-i)} = \frac{-(wi+i^2)}{(w-i)} = \frac{1-wi}{w-i}$$

$$\frac{z+1}{z-1} = \frac{1-wi}{w-i}$$

$$\frac{z+\cancel{1}+z-\cancel{1}}{z+1-(z-1)} = \frac{1-wi+w-i}{1-wi-w+i}$$

$$\frac{\cancel{2}z}{\cancel{2}} = \frac{1-wi+w-i}{1-wi-w+i}$$

$$z = \frac{(1+w)-i(w+1)}{(1-w)+i(-w+1)} = \frac{(w+1)(1-i)}{(1-w)(1+i)}$$

$$\frac{w+i}{w-i} = \left(\frac{z+1}{z-1}\right)\frac{(1-i)(1-i)}{(i+1)(-i+1)}$$

$$\frac{w+i}{w-i} = \left(\frac{z+1}{z-1}\right)\cdot\frac{1+i^2-2i}{1^2-i^2}$$

$$\frac{w+i}{w-i} = \left(\frac{z+1}{z-1}\right)\cdot\frac{\cancel{1}-\cancel{1}-2i}{2} = \frac{(z+1)}{(z-1)}(-i)$$

$$\frac{w+i}{w-i} = \frac{-iz-i}{(z-1)}$$

$$\frac{w+i+w-i}{w+i-(w-i)} = \frac{-iz-i+z-1}{-iz-i-z+1}$$

$$\frac{\cancel{2}w}{\cancel{2}i} = \frac{-iz-i+z-1}{-iz-i-z+1}$$

$$w = \frac{-i^2z+i^2+iz-i}{-iz-i-z+1} = \frac{z+1+iz-i}{-z+1-iz-i}$$

$$w = \frac{z+1+i(z-1)}{(1-z)+i(z+1)}$$

$$\frac{w+i}{w-i} = \frac{(1-i)}{(i+1)}\frac{(1+z)}{(z-1)} = \frac{1+z-i-iz}{iz-i+z-1}$$

$$\frac{(w+i)+(w-i)}{(w+i)-(w-i)} = \frac{\cancel{1}+z-i-\cancel{iz}+\cancel{iz}-i+z-\cancel{1}}{1+\cancel{z}-\cancel{i}-iz-iz+\cancel{i}-\cancel{z}+1}$$

$$\frac{\cancel{2}w}{\cancel{2}i} = \frac{2z-2i}{-2iz+2}$$

$$\frac{w}{i} = \frac{\cancel{2}(z-i)}{\cancel{2}(-iz+1)}$$

$$w = \left(\frac{iz-i^2}{-iz+1}\right) = \left(\frac{iz+1}{-iz+1}\right).$$

first part

2nd part:-

$$\frac{w}{1} = \frac{iz+1}{-iz+1}$$

$$-izw + w = iz+1$$

$$w-1 = iz+izw = z(iw+i)$$

$$\boxed{z = \frac{w-1}{i(w+1)}}$$

For $|z|<1 \Rightarrow \left|\frac{w-1}{i(w+1)}\right| < 1$

$$\Rightarrow \quad |w-1| < |w+1||i|$$

$$\Rightarrow \quad |(u-1)+iv| < |(u+1)+iv|$$

$$\Rightarrow \quad \sqrt{(u-1)^2+v^2} < \sqrt{(u+1)^2+v^2}$$

Squaring

$$\Rightarrow \quad (u-1)^2+v^2 < (u+1)^2+v^2$$

$$\Rightarrow \quad \cancel{u^2}+\cancel{1}-2u+\cancel{v^2} < \cancel{u^2}+\cancel{1}+2u+\cancel{v^2}$$

$$\Rightarrow \quad -2u < 2u$$

$$\Rightarrow \quad 2u+2u > 0$$

$$\Rightarrow \quad \boxed{u > 0}$$

Que2 Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$. Also find the image of unit circle.

Sol:- Given $z_1 = 0, \quad z_2 = -1, \quad z_3 = i, \quad z_4 = z$

$w_1 = i, \quad w_2 = 0, \quad w_3 = \infty, \quad w_4 = w$

By cross Ratio

$$\frac{(z_1-z_2)}{(z_2-z_3)}\frac{(z_3-z_4)}{(z_4-z_1)} = \frac{(w_1-w_2)}{(w_2-w_3)}\frac{(w_3-w_4)}{(w_4-w_1)}$$

$$\Rightarrow \quad \frac{(0+1)}{(-1-i)}\frac{(i-z)}{(z-0)} = \frac{(i-0)}{(0-\infty)}\frac{(\infty-w)}{(w-i)}$$

$$\Rightarrow \quad \frac{i-z}{-z-zi} = \frac{i}{(w-i)}(-1) = \frac{-i}{w-i}$$

$$\Rightarrow \quad \frac{w-i}{-i} = \frac{-z-zi}{i-z}$$

$$\Rightarrow \quad w-i = \frac{iz+zi^2}{(-z+i)}$$

$$\Rightarrow \quad w = \frac{iz+zi^2}{(-z+i)} + i = \frac{\cancel{iz}-z-\cancel{iz}+i^2}{(-z+i)}$$

$$\Rightarrow \quad w = \frac{+(z+1)}{+(z-i)} = \left(\frac{z+1}{z-i}\right) \checkmark$$

2nd part:-

$$w = \left(\frac{z+1}{z-i}\right)$$

$$\Rightarrow \quad z = \left(\frac{iw+1}{w-1}\right) \qquad \left(\because z = \frac{-dw+b}{cw-a}\right)$$

from $|z| = 1 \Rightarrow |iw+1| = |w-1|$

$\Rightarrow \quad |1+i(u+iv)| = |u+iv-1|$

$\Rightarrow \quad |(1-v)+iu| = |(u-1)+iv|$

$\Rightarrow \quad (1-v)^2+u^2 = (u-1)^2+v^2$

$\Rightarrow \quad 1+v^2-2v+u^2 = u^2+1-2u+v^2$

$\Rightarrow \quad \boxed{u=v}$

Que 3] Find the fixed point of the transformation

$$w = \frac{2z-5}{z+4}.$$

Sol:- Here $w = f(z) = \dfrac{2z-5}{z+4}$.

for fixed point $\quad f(z)=z \Rightarrow \dfrac{2z-5}{z+4} = z$

$\Rightarrow \quad z^2+2z+5=0$

$\Rightarrow \quad z = \dfrac{-2 \pm \sqrt{4-2\times1\times5}}{2\times1} = \underline{\underline{-1 \pm 2i}}$

Que 4] Show that the transformation $w = \left(\dfrac{5-4z}{4z-2}\right)$ transform the circle $|z|=1$ into a circle of radius unity in w-plane and find the centre of circle

Sol:- Here $\quad w = \left(\dfrac{5-4z}{4z-2}\right) \quad \Rightarrow z = \left(\dfrac{2w+5}{4w+4}\right)$

Now from $|z|=1$

$\Rightarrow \quad \left|\dfrac{2w+5}{4w+4}\right| = 1$

$\Rightarrow \quad |2w+5| = |4+4w|$

$\Rightarrow \quad |2u+2iv+5| = |4+4u+iv4|$

$\Rightarrow \quad (2u+5)^2+(2v)^2 = (4u+4)^2+(4v)^2$

$\Rightarrow \quad 4u^2+25+20u+4v^2 = 16u^2+16+32u+16v^2$

$\Rightarrow \quad 12u^2+12v^2+12u-9=0$

$\Rightarrow \quad \boxed{u^2+v^2+u-\frac{3}{4}=0}$

Which is the eq. of circle in w-plane.

Comparing with $\quad u^2+v^2+2gu+2fv+c=0$

$\therefore \quad g=\frac{1}{2}, \quad f=0, \quad c=-\frac{3}{4}$

Centre $= (-g,-f) = (-\frac{1}{2},0)$, $\qquad$ Radius $= \sqrt{g^2+f^2-c} = \sqrt{\frac{1}{4}+0+\frac{3}{4}} = 1$.