Module-5: Vector Calculus ①

Scalar and vector functions

Scalar function $f(x,y,z)$ is a function defined at each point in a certain domain D in a space. Its value is real and depends only on the point $P(x,y,z)$ in space but not on any particular coordinate system being used.

If to each value of a scalar variable t, there corresponds a value of a vector $\vec{r}$, then $\vec{r}$ is called a vector function of the scalar variable t and we write $\vec{r} = \vec{r}(t)$ or $\vec{r} = \vec{f}(t)$.

For eg, the position vector $\vec{r}$ of a particle moving along a curved path is a vector function of time scalar.

$\vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ denote unit vectors along the axis of x, y, z respectively and $f_1(t), f_2(t), f_3(t)$ are called the components of the vector $\overrightarrow{f(t)}$ along the co-ordinate axes.

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Derivative of a vector function with respect to a Scalar

Let $\vec{r} = \vec{f}(t)$ be a vector function of the Scalar variable t.

then $\frac{d\vec{r}}{dt}$ is itself a vector function of t.

and its derivative is denoted by $\frac{d^2\vec{r}}{dt^2}$ and is called the second derivative of $\vec{r}$ with respect to t.

<u>Note</u> $\hat{i}$, $\hat{j}$ $\hat{k}$ being fixed Unit vectors are Constant vectors.

$$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \vec{0}$$

<u>Qu</u> find the Unit tangent vector at any Point on the Curve $x = t^2+2$, $y = 4t-5$, $z = 2t^2-6t$, where t is any variable. Also determine the Unit tangent vector at the point $t=2$

<u>Sol</u> If $\vec{r}$ is the position vector of any Point (x, y, z) on the given Curve then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = (t^2+2)\hat{i} + (4t-5)\hat{j} + (2t^2-6t)\hat{k}$$

The vector $\frac{d\vec{r}}{dt}$ is along the tangent at the point (x, y, z) to the given Curve.

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Examples. A particle moves on the curve $x = 2t^2, y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of velocity and accelaration on time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$

Soln If $\vec{r}$ is the Position vector of any point (x, y, z) on the given Curve, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t-5)\hat{k}$$

Velocity $\vec{V} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t-4)\hat{j} + 3\hat{k}$

$$= 4\hat{i} - 2\hat{j} + 3\hat{k} \text{ at } t = 1$$

Accelaration $\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\hat{i} + 2\hat{j}$ at $t = 1$

Now Unit vector in the given direction

$$\hat{i} - 3\hat{j} + 2\hat{k} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}} = \hat{n}$$

The Component of velocity in the given direction

$$= \vec{V} \cdot \hat{n} = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}}$$

$$= \frac{4(1) - 2(-3) + 3(2)}{\sqrt{14}} = \frac{16\sqrt{14}}{14} = \frac{8\sqrt{14}}{7}$$

and the Component of accelaration in the given direction

$$= \vec{a} \cdot \hat{n} = (4\hat{i} + 2\hat{j}) \cdot \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}}$$

$$= \frac{-2}{\sqrt{14}} = \frac{-\sqrt{14}}{7}$$

Now $\frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$ (3)

and $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2}$

$= 2\sqrt{5t^2 - 12t + 13}$

The Unit tangent vector $\hat{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{t\hat{i} + 2\hat{j} + (2t-3)\hat{k}}{\sqrt{5t^2 - 12t + 13}}$

Also Unit tangent vector at that point $t = 2$

is $\frac{2\hat{i} + 2\hat{j} + (2\times2-3)\hat{k}}{\sqrt{5\times4 - 12\times2 + 13}} = \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$

Q4 Find the angle between the tangents to the curve $\vec{r} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$ at the point $t = \pm1$

Soln $\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} + 3t^2\hat{k}$ is a vector along the tangent at any point 't'

If T_1 and T_2 are the vectors along the tangents at $t=1$ and $t=-1$ respectively, then

$\vec{T_1} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{T_2} = -2\hat{i} + 2\hat{j} + 3\hat{k}$

If θ is the angle between $\vec{T_1}$ and $\vec{T_2}$. then

$$\cos\theta = \frac{\vec{T_1}\cdot\vec{T_2}}{|\vec{T_1}||\vec{T_2}|} = \frac{2(-2) + 2(2) + 3(3)}{\sqrt{4+4+9}\sqrt{4+4+9}} = \frac{9}{17}$$

$$\theta = \cos^{-1}\left(\frac{9}{17}\right)$$

Scalar and vector fields

A variable quantity whose value at any point in a region of space depends upon the position of the point. There are two ~~types~~ types of point function

Scalar Point function Let R be a region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given then ϕ is called scalar function and R is called a scalar field.

The temperature distribution in a medium, the distribution of atmospheric pressure in space are examples of scalar point function.

Vector Point function :- Let R be a region of space at each point of which a vector $\vec{V} = \vec{V}(x, y, z)$ is given, then $\vec{V}$ is called a vector point function and R is called a vector field.

Every vector $\vec{V}$ of the field is regarded as a localised vector attached to the corresponding point (x, y, z)

The velocity of a moving fluid at any instant the gravitational force are example of vector point function

Gradient of a Scalar field

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Let $\phi(x,y,z)$ be function defining a Scalar field. the the vector $\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$ is Called gradient of Scalar field ϕ and is denoted as grad ϕ

Thus $$\boxed{\text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}}$$

The gradient of Scalar fied ϕ is obtained by operating on ϕ by the veccotr operator $\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$

$$\text{del}\,(\nabla) = \hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

$$\text{grad}\,\phi = \nabla\phi$$

Note Gradient of a Scalar field ϕ is a vector normal to the Surface $\phi=c$ and has a magnitude equal to the rate of change of ϕ along this normal

$$|\nabla\phi| = \frac{\partial\phi}{\partial n}$$

Directional derivative: Directional derivative of a Scalar field f at a Point $P(x,y,z)$ in the direction of a Unit vector $\hat{a}$ is given by

$$\boxed{\frac{\partial f}{\partial s} = (\text{grad}\,f).\hat{a}}$$

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* Find grad ϕ when $\phi = 3x^2y - y^3z^2$ at point $(1,-2,-1)$

Sol^n grad $\phi = \nabla\phi = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k} \text{ at point } (1,-2,-1)$$

* Show that $\nabla r^n = n r^{n-2}\vec{r}$ and hence evaluate $\nabla \frac{1}{r}$ when $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol^n $\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$

$$= \hat{i}\left(n r^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(n r^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(n r^{n-1}\frac{\partial r}{\partial z}\right) \quad (1)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2 \quad (2)$$

Diff (2) partially w.r.t 'x'

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

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from ① $\nabla r^n = \hat{i}\left(n r^{n-1} \cdot \frac{x}{r}\right) + \hat{j}\left(n r^{n-1} \cdot \frac{y}{r}\right) + \hat{k}\left(n r^{n-1} \cdot \frac{z}{r}\right)$

$= n r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$

$= n r^{n-2}\, \vec{r}$ —③

Putting $n=-1$ in ③ we get

$$\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$$

Ex If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, Prove that grad u, grad v, grad w are Coplanar Vectors.

Soln

$$\text{grad}\,u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+y+z) = \hat{i}+\hat{j}+\hat{k}$$

$$\text{grad}\,v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad}\,w = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz+zx+xy)$$

$$= \hat{i}(z+y) + \hat{j}(z+x) + \hat{k}(y+x)$$

$$\text{grad}\,u \cdot (\text{grad}\,v \times \text{grad}\,w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ z+y & z+x & y+x \end{vmatrix}$$

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$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & y+z+x & z+y+x \\ z+y & z+x & y+x \end{vmatrix} \quad (R_2 \to R_2 + R_3)$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

Hence grad u, grad v, grad w are Coplanar vectors

* Show that $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ is constant vector.

Soln $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 Constant

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

$$\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$= \vec{a}$$

* Find a Unit ~~nor~~ vector normal to the surface $x^3+y^3+3xyz=3$ at $(1,2,-1)$

Soln $\phi = x^3+y^3+3xyz-3$

$$\frac{\partial\phi}{\partial x} = 3x^2+3yz, \quad \frac{\partial\phi}{\partial y} = 3y^2+3xz, \quad \frac{\partial\phi}{\partial z} = 3xy$$

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$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (3x^2+3yz)\hat{i} + (3y^2+3xz)\hat{j} + (3xy)\hat{k}$$

At $(1,2,-1)$, $\nabla\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$

Which a ~~unit~~ vector normal to the given surface at $(1,2,-1)$.

Hence a unit ~~no~~ vector normal to the given surface at $(1,2,-1)$

$$= \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{(-3)^2+(9)^2+(6)^2}} = \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{3\sqrt{14}}$$

$$= \frac{1}{\sqrt{14}}(-\hat{i}+3\hat{j}+2\hat{k})$$

(11) Find a unit vector normal to the surface $x^2y + 2xz = 4$ at point $(2,-2,3)$

Let $\phi = x^2y + 2xz - 4$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + 2xz - 4)$$

$$= (2xy+2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

$$= -2\hat{i} + 4\hat{j} + 4\hat{k} \text{ at } (2,-2,3)$$

$$|\text{grad}\,\phi| = \sqrt{4+16+16} = 6$$

$$\text{Unit normal vector} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{-\hat{i}+2\hat{j}+2\hat{k}}{3}.$$

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* Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at point $(2,-1,2)$

Sol^n Angle between two surface at a point is the angle between the normals to the surfaces at that point.

Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2 = x^2+y^2-z-3=0$

Then $\text{grad}\,\phi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$ and $\text{grad}\,\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$

Let $\vec{n}_1 = \text{grad}\,\phi_1$ at the point $(2,-1,2)$

$$n_1 = 4\hat{i}-2\hat{j}+4\hat{k}$$

$n_2 = \text{grad}\,\phi_2$ at point $(2,-1,2)$

$$n_2 = 4\hat{i}-2\hat{j}-\hat{k}$$

The vectors $\vec{n}_1$ and $\vec{n}_2$ are along normals to the two surface at $(2,-1,2)$. If θ is the angle between these vectors, then

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|.|\vec{n}_2|}$$

$$= \frac{4(4)-2(-2)+4(-1)}{\sqrt{16+4+16}\,\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

* Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of line PQ where Q is the point $(5,0,4)$.
In what direction it will be max? Find also the magnitude of this max.

Sol^n We have $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

$$= 2\hat{i} - 4\hat{j} + 12\hat{k} \text{ at } P(1,2,3)$$

Also $\vec{PQ} = \vec{OQ} - \vec{OP}$

$$= (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 4\hat{i} - 2\hat{j} + \hat{k}$$

If $\hat{n}$ is a unit vector in the direction $\vec{PQ}$

then $\hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

Directional derivative of f in the direction $\vec{PQ}$

$$= (\nabla f)\cdot\hat{n}$$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k})\cdot\frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$$

$$= \frac{1}{\sqrt{21}}\left[2(4) - 4(-2) + 12(1)\right]$$

$$= \frac{28}{\sqrt{21}}$$

The directional derivative of f is maximum in the direction of the normal to the given surface ie in the direction of $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

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The maximum value of this directional derivative $= |\nabla f|$

$$= \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164}$$

** What is the greatest rate of increase of $u = xyz^2$ at point $(1, 0, 3)$?

$$u = xyz^2$$

$$\text{grad } u = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}$$

$$= yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$$

$$= 9\hat{j} \text{ at } (1, 0, 3)$$

** If $\nabla\phi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$

find ϕ

Sol^n Let $\vec{F} = \nabla\phi$

$$\Rightarrow \vec{F} \cdot d\vec{a} = \nabla\phi \cdot d\vec{a}$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$d\phi = \vec{F} \cdot d\vec{a}$$

$$= [(y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}][dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$= (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)dz$$

$$= (y^2dx + 2xy\,dy) - (2xyz^3dx + x^2z^3dy + 3x^2yz^2dz) + 3dy + 6z^3dz$$

$$= d(xy^2) - d(x^2yz^3) + d(3y) + d\left(\frac{3}{2}z^4\right)$$

$$\phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + C$$

from ①

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$$(8\lambda+10\lambda)^2+(4\lambda-22\lambda)^2+(-11\lambda+20\lambda)^2=225$$

$$\lambda=\pm\frac{5}{9}$$

$$a=\pm\frac{20}{9},\quad b=\pm\frac{55}{9},\quad c=\pm\frac{50}{9}$$

xxx Find the directional derivative of $\phi=5x^2y-5y^2z+\frac{5}{2}z^2x$ at point $P(1,1,1)$ in the direction of line

$$\frac{x-1}{2}=\frac{y-3}{-2}=\frac{z}{1}$$

Sol^n $\phi=5x^2y-5y^2z+\frac{5}{2}z^2x.$

$$\text{grad}\,\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

$$=\left(10xy+\frac{5}{2}z^2\right)\hat{i}+(5x^2-10yz)\hat{j}+(-5y^2+5zx)\hat{k}$$

$$=\frac{25}{2}\hat{i}-5\hat{j}\text{ at }(1,1,1)$$

Here $\hat{a}=\dfrac{2\hat{i}-2\hat{j}+\hat{k}}{3}$

Directional derivative $=(\text{grad}\,\phi)\cdot\hat{a}$

$$=\left(\frac{25}{2}\hat{i}-5\hat{j}\right)\cdot\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}\right)$$

$$=\frac{25}{3}+\frac{10}{3}=\frac{35}{3}$$

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* If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the value of a, b and c.

Soln $\quad \phi = ax^2y + by^2z + cz^2x$

$$\text{grad}\,\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)\hat{k}$$

$$= (2a+c)\hat{i} + (a+2b)\hat{j} + (b+2c)\hat{k} \text{ at } (1,1,1)$$

Now directional derivative ~~at~~ is maximum along the normal to the surface ie along grad ϕ

$$|\text{grad}\,\phi| = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2}$$

$$15 = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2}$$

$$(2a+c)^2 + (a+2b)^2 + (b+2c)^2 = 225 \quad \text{--- (1)}$$

But we are given that directional derivative is maximum in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ ie parallel to vector $2\hat{i} - 2\hat{j} + \hat{k}$ Hence,

$$\frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{2c+b}{1}$$

$$\Rightarrow 2a + c = -a - 2b \Rightarrow 3a + 2b + c = 0 \quad \text{--- (2)}$$

$$\Rightarrow 2b + a = -4c - 2b \Rightarrow a + 4b + 4c = 0 \quad \text{--- (3)}$$

By cross-multiplication we get

$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = \lambda \text{ (say)}$$

$$a = 4\lambda,\ b = -11\lambda,\ c = 10\lambda \quad \text{--- (4)}$$

Gradient in Polar co-ordinates

$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta}$$

Example If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that

(i) grad $r = \frac{\vec{r}}{r}$ (ii) grad $\frac{1}{r} = \frac{-\vec{r}}{r^3}$

(iii) grad $r^n = n r^{n-2}\vec{r}$ where $r = |\vec{r}|$

Soln (i) grad $r = \frac{\partial(r)}{\partial r}\hat{r} = \hat{r} = \frac{\vec{r}}{r}$

(ii) grad $\frac{1}{r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r} = \frac{-1}{r^2}\hat{r} = \frac{-\vec{r}}{r^3}$

(iii) grad $r^n = \frac{\partial}{\partial r}(r^n)\hat{r} = n r^{n-1}\hat{r} = n r^{n-1}\frac{\vec{r}}{r} = n r^{n-2}\vec{r}$

Where $\hat{r}$ is the unit vector in the direction of $\vec{r}$

Example Find $\nabla|\vec{r}|^2$

Soln $\nabla|\vec{r}|^2 = \nabla r^2 = \frac{\partial}{\partial r}(r^2)\hat{r} = 2r\hat{r} = 2r\frac{\vec{r}}{|\vec{r}|}$

$= 2\vec{r}$.

Ex Evaluate grad e^{r^2}

$\nabla e^{r^2} = \frac{\partial}{\partial r}(e^{r^2})\hat{r}$

$= e^{r^2}\cdot 2r\,\hat{r}$

$= 2e^{r^2} r\frac{\vec{r}}{|\vec{r}|}$

$= 2e^{r^2}\vec{r}$.

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Ex Find the directional derivative of $\frac{1}{r^2}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(II) Find the directional derivative of $\frac{1}{r^n}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol^n (I) $\nabla\left(\frac{1}{r^2}\right) = \frac{-2}{r^3}\hat{r} = \frac{-2}{r^4}\vec{r}$

Let $\hat{a}$ be the Unit vector in the direction of $\vec{r}$ then $\hat{a} = \hat{r} = \frac{\vec{r}}{r}$

Directional derivative $= -\frac{2}{r^4}\vec{r} \cdot \frac{\vec{r}}{r} = \frac{-2}{r^5}(r^2) = \frac{-2}{r^3}$

(II) $\nabla\left(\frac{1}{r^n}\right) = \frac{-n}{r^{n+1}}\hat{r} = \frac{-n}{r^{n+2}}\vec{r}$

$\hat{a} = \hat{r} = \frac{\vec{r}}{r}$

Directional derivative $= \nabla\left(\frac{1}{r^n}\right) \cdot \hat{a}$

$= \frac{-n}{r^{n+2}}\vec{r} \cdot \frac{\vec{r}}{r}$

$= \frac{n}{r^{n+3}}(r^2) = \frac{-n}{r^{n+1}}$

** Show that grad $f(r) \times \vec{r} = \vec{0}$ (17)

$$\text{grad } f(r) = \frac{\partial}{\partial r}\{f(r)\}\,\hat{r}$$

$$= f'(r)\,\hat{r} = f'(r)\frac{\vec{r}}{r}.$$

$$\text{grad } f(r) \times \vec{r} = \frac{f'(r)}{r}(\vec{r}\times\vec{r}) = \vec{0}$$

** If $f = r\cos\theta + \tan\theta$ find grad f in polar coordinates

$$\text{grad } f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta$$

$$= \frac{\partial}{\partial r}(r\cos\theta + \tan\theta)\,\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}(r\cos\theta + \tan\theta)\hat{e}_\theta$$

$$= \cos\theta\,\hat{e}_r + \frac{1}{r}(-r\sin\theta + \sec^2\theta)\,\hat{e}_\theta$$

** Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol $\phi = \frac{1}{r}$

$$\nabla\phi = \nabla\frac{1}{r} = \frac{-1}{r^2}\hat{r} = -\frac{\vec{r}}{r^3}$$

Let $\hat{a}$ be the Unit vector in the direction of $\vec{r}$

$$\hat{a} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

directional derivative $= \nabla\phi\cdot\hat{a}$

$$= \left(\frac{-\vec{r}}{r^3}\right)\cdot\frac{\vec{r}}{r} = \frac{-1}{r^4}(\vec{r}\cdot\vec{r})$$

$$= \frac{-r^2}{r^4} = \frac{-1}{r^2}$$

Divergence of a vector point function

The divergence of a differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \qquad \left[\begin{array}{l}\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0\end{array}\right]$$

Curl of a vector point function

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{Curl}\,\vec{V} = \nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

<u>Example</u> If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, Show that

(i) $\text{div}\,\vec{r} = 3$ (II) $\text{curl}\,\vec{r} = 0$

<u>Solⁿ</u> (i) $\text{div}\,\vec{r} = \nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

(II) $\text{curl}\,\vec{r} = \nabla\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= \vec{0}$$

<u>Example</u> Find the divergence and curl of the vector $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at point $(2, -1, 1)$

<u>Solⁿ</u> $\text{div}\,\vec{V} = \nabla\cdot\vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left((xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}\right)$$

$$= yz + 3x^2 + 2xz - y^2 = -1 + 12 + 4 - 1 = 14 \text{ at } (2,-1,1)$$

$\text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$

$= \hat{i}(-2yz-0)+\hat{j}(xy-z^2)+\hat{k}(6xy-xz)$ (21)

$= 2\hat{i} - 3\hat{j} - 14\hat{k}$ at $(+2,+1,-1)$

Example:- Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

Soln $\phi = x^3+y^3+z^3-3xyz$, then

$$\vec{F} = \text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$=(3x^2-3yz)\hat{i}+(3y^2-3zx)\hat{j}+(3z^2-3xy)\hat{k}$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(3x^2-3yz)\hat{i}+(3y^2-3zx)\hat{j} + (3z^2-3xy)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3zx)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x+6y+6z = 6(x+y+z)$$

and

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3zx & 3z^2-3xy \end{vmatrix}$$

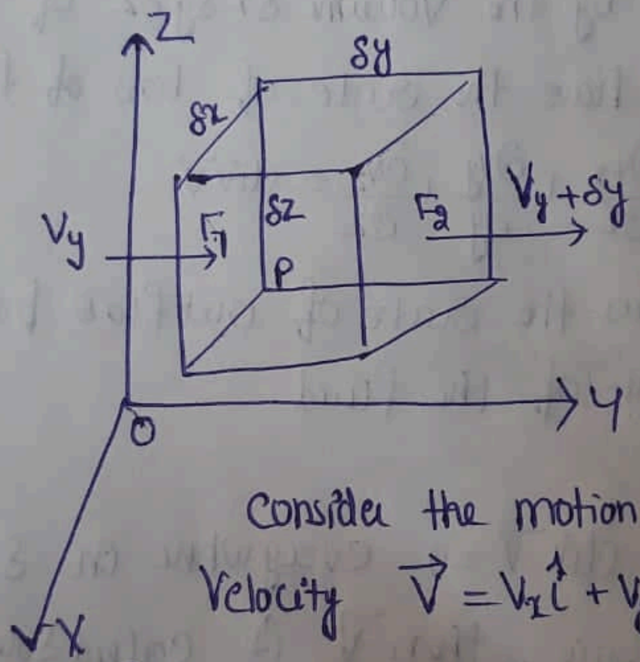
$$= \hat{i}(-3x+3x)+\hat{j}(-3y+3y)+\hat{k}(-3z+3z) = \vec{0}$$

Physical Interpretation of Divergence

Consider the fluid having density $\rho = \rho(x,y,z,t)$ and velocity $\vec{v} = \vec{v}(x,y,z,t)$ at a point (x,y,z) at time t.

Let $\vec{V} = \rho\vec{v}$, then $\vec{V}$ is a vector having the same direction as $\vec{v}$ and magnitude $\rho|\vec{v}|$. It is known flux.

Its direction gives the direction of the fluid flow and its magnitude gives the mass of the fluid crossing per unit time a unit area placed perpendicular to the direction of flow



Consider the motion of the fluid having velocity $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ at a point $P(x,y,z)$. Consider a small parallelopiped with edges $\delta x, \delta y, \delta z$ parallel to the axes with one of its corner at P.

The mass of fluid entering through the face F_1 per unit time is $V_y\delta x\,\delta z$ and that flowing out through the opposite face F_2 is $V_{y+\delta y}\delta x\,\delta z = \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\,\delta z$

by Taylor's Series.

The net decrease in the mass of fluid flowing across these two faces

$$\left(V_y + \frac{\partial v_y}{\partial y}\delta y\right) - V_y \delta x \delta z = \frac{\partial v_y}{\partial y}\delta x \delta y \delta z$$

Similarly, Considering the other two pairs of faces, we get the total decrease in the mass of fluid inside the parallelopiped per unit time $= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta x \delta y \delta z$

Dividing this by the volume $\delta x \delta y \delta z$ of the parallelopiped, we have the rate of loss of fluid per unit time $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \text{div}\,\vec{v}$.

Hence $\text{div}\,\vec{v}$ gives the rate of outflow per unit volume at a point of the fluid.

Note If $\text{div}\,\vec{v} = 0$ everywhere in some region R of space, then $\vec{v}$ is called Solenoidal vector point function.

Physical Interpretation of curl

Consider a rigid body rotating about a fixed axis through O with uniform angular velocity

$$\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$$

The velocity $\vec{V}$ of any point $P(x, y, z)$ on the body is given by $\vec{V} = \vec{\omega} \times \vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of P.

$$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & k \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}$$

$$= 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) = 2\vec{\omega}$$

$$\boxed{\vec{\omega} = \frac{1}{2}\text{curl } \vec{V}}$$

Thus, the angular velocity at any point is equal to the half the curl of the linear velocity at that point of the body.

Note If $\text{curl } \vec{V} = \vec{0}$, then $\vec{V}$ is called irrotational vector.

for constant vector $\vec{a}$, $\text{div } \vec{a} = 0$

$\text{curl } \vec{a} = 0$

Vector Identities

1- $\boxed{div(grad\ \phi) = \nabla^2 \phi}$

Proof $div(grad\ \phi) = \nabla \cdot (\nabla \phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2 \phi$$

2- $\boxed{Curl(grad\ \phi) = \nabla \times \nabla \phi = \vec{0}}$

Proof $Curl(grad\ \phi) = \nabla \times \nabla \phi$

$$= \nabla \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right] - \hat{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right] + \hat{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

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③ $\boxed{\operatorname{div}(\operatorname{curl}\vec{v}) = \nabla\cdot(\nabla\times\vec{v}) = 0}$

Proof let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then

$$\operatorname{curl}\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\operatorname{div}(\operatorname{curl}\vec{v}) = \nabla\cdot(\nabla\times\vec{v})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 v_3}{\partial x\,\partial y} - \frac{\partial^2 v_2}{\partial x\,\partial z}\right) + \left(\frac{\partial^2 v_1}{\partial y\,\partial z} - \frac{\partial^2 v_3}{\partial y\,\partial x}\right) + \left(\frac{\partial^2 v_2}{\partial z\,\partial x} - \frac{\partial^2 v_1}{\partial z\,\partial y}\right) = 0$$

* ④ If $\vec{a}$ is a vector function and u is a scalar functi
then $\boxed{\operatorname{div}(u\vec{a}) = u\operatorname{div}\vec{a} + (\operatorname{grad} u)\cdot\vec{a}}$

Proof

$$\operatorname{div}(u\vec{a}) = \sum \hat{i}\cdot\frac{\partial}{\partial x}(u\vec{a})$$

$$= \sum \hat{i}\cdot\left(u\frac{\partial \vec{a}}{\partial x} + \frac{\partial u}{\partial x}\vec{a}\right)$$

$$= u\sum \hat{i}\cdot\frac{\partial \vec{a}}{\partial x} + \left(\sum i\frac{\partial u}{\partial x}\right)\cdot\vec{a}$$

$$= u\operatorname{div}\vec{a} + (\operatorname{grad} u)\cdot\vec{a}$$

(5) If $\vec{a}$ is a vector function and u is a scalar function then

$$\boxed{\text{Curl}(u\vec{a}) = u\,\text{Curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}}$$

Proof $\text{Curl}(u\vec{a}) = \sum \hat{i}\times\frac{\partial}{\partial x}(u\vec{a})$

$$= \sum \hat{i}\times\left(u\frac{\partial\vec{a}}{\partial x} + \frac{\partial u}{\partial x}\vec{a}\right)$$

$$= u\left(\sum \hat{i}\times\frac{\partial\vec{a}}{\partial x}\right) + \sum \hat{i}\frac{\partial u}{\partial x}\times\vec{a}$$

$$= u\,\text{Curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}$$

(6) $\text{div}(\vec{a}\times\vec{B}) = \vec{B}\cdot\text{Curl}\,\vec{a} - \vec{a}\cdot\text{Curl}\,\vec{B}$

Proof $\text{div}(\vec{a}\times\vec{B}) = \sum \hat{i}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{b})$

$$= \sum \hat{i}\cdot\left(\frac{\partial\vec{a}}{\partial x}\times\vec{B} + \vec{a}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum \hat{i}\cdot\left(\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right) - \sum \hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{a}\right)$$

$$= \sum\left(\hat{i}\times\frac{\partial\vec{a}}{\partial x}\right)\cdot\vec{b} - \sum\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{a}$$

$$= (\text{Curl}\,\vec{a})\cdot\vec{B} - (\text{Curl}\,\vec{B})\cdot\vec{a}$$

$$= \vec{B}\cdot\text{Curl}\,\vec{a} - \vec{a}\cdot\text{Curl}\,\vec{B}$$

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7) $\text{Curl}(\vec{a}\times\vec{b}) = \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a} + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$

Proof $\text{Curl}(\vec{a}\times\vec{b}) = \sum \hat{i}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{b})$

$$= \sum \hat{i}\times\left(\frac{\partial \vec{a}}{\partial x}\times\vec{b} + \vec{a}\times\frac{\partial \vec{b}}{\partial x}\right)$$

$$= \sum \hat{i}\times\left(\frac{\partial \vec{a}}{\partial x}\times\vec{b}\right) + \sum \hat{i}\times\left(\vec{a}\times\frac{\partial \vec{b}}{\partial x}\right)$$

$$= \sum\left[(\hat{i}\cdot\vec{b})\frac{\partial \vec{a}}{\partial x} - \left(\hat{i}\cdot\frac{\partial \vec{a}}{\partial x}\right)\vec{b}\right] + \sum\left[\left(\hat{i}\cdot\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - (\hat{i}\cdot\vec{a})\frac{\partial \vec{b}}{\partial x}\right]$$

$$\boxed{\text{Formula} \quad \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}}$$

$$= \sum(\vec{b}\cdot\hat{i})\frac{\partial \vec{a}}{\partial x} - \left(\sum \hat{i}\cdot\frac{\partial \vec{a}}{\partial x}\right)\vec{b} + \sum\left(\hat{i}\cdot\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - \sum(\vec{a}\cdot\hat{i})\frac{\partial \vec{b}}{\partial x}$$

$$= \left(\vec{b}\cdot\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{a} - \vec{b}\,\text{div}\,\vec{a} + \vec{a}\,\text{div}\,\vec{b} - \left(\vec{a}\cdot\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{b}$$

$$= (\vec{b}\cdot\nabla)\vec{a} - \vec{b}\,\text{div}\,\vec{a} + \vec{a}\,\text{div}\,\vec{b} - (\vec{a}\cdot\nabla)\vec{b}$$

$$= \vec{a}\,\text{div}\,\vec{b} - \vec{b}\,\text{div}\,\vec{a} + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$$

(29)

(8) : $\text{Curl}(\text{Curl}\vec{V}) = \text{grad}(\text{div}\vec{V}) - \nabla^2\vec{V}$

Proof. Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then $\text{Curl}\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore \text{Curl}(\text{Curl}\vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} & \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} & \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{vmatrix}$

$$= \sum\hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right]$$

$$= \sum\hat{i}\left[\frac{\partial^2 V_2}{\partial y\,\partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} + \frac{\partial^2 V_3}{\partial x\,\partial z}\right]$$

$$= \sum\hat{i}\left[\frac{\partial^2 V_2}{\partial x\,\partial y} + \frac{\partial^2 V_3}{\partial x\,\partial z} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} + \frac{\partial^2 V_1}{\partial x^2} - \frac{\partial^2 V_1}{\partial x^2}\right]$$

from your side.

$$= \sum\hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right]$$

$$= \sum\hat{i}\left[\frac{\partial}{\partial x}(\nabla\cdot\vec{V}) - (\nabla^2 V_1)\right]$$

$$= \sum\hat{i}\frac{\partial}{\partial x}(\nabla\cdot\vec{V}) - \nabla^2\sum\hat{i}V_1 \quad = \nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V}$$

$$= \text{grad}(\text{div}\vec{V}) - \nabla^2\vec{V}$$

(9) $\text{grad}\,(\vec{a}\cdot\vec{b}) = \vec{a}\times \text{curl}\,\vec{b} + \vec{b}\times\text{curl}\,\vec{a} + (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a}$

(80)

<u>Proof</u>

$$\text{grad}\,(\vec{a}\cdot\vec{b}) = \Sigma\,\hat{i}\,\frac{\partial}{\partial x}(\vec{a}\cdot\vec{b})$$

$$= \Sigma\,\hat{i}\left[\vec{a}\cdot\frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x}\cdot\vec{b}\right]$$

$$= \Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)\hat{i} + \Sigma\left(\vec{b}\cdot\frac{\partial \vec{a}}{\partial x}\right)\hat{i} \quad \text{— (1)}$$

We know that

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$(\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$$

$$\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)\hat{i} = (\vec{a}\cdot\hat{i})\frac{\partial \vec{b}}{\partial x} - \vec{a}\times\left(\frac{\partial \vec{b}}{\partial x}\times\hat{i}\right)$$

$$\cancel{= (\vec{a}\cdot\hat{i})\frac{\partial \vec{b}}{\partial x} + \vec{a}\times}$$

$$= (\vec{a}\cdot\hat{i})\frac{\partial \vec{b}}{\partial x} + \vec{a}\times\left(\hat{i}\times\frac{\partial \vec{b}}{\partial x}\right)$$

$$\Sigma\left(\vec{a}\cdot\frac{\partial \vec{b}}{\partial x}\right)\hat{i} = \left(\vec{a}\cdot\Sigma\,\hat{i}\frac{\partial}{\partial x}\right)\vec{b} + \vec{a}\times\Sigma\left(\hat{i}\times\frac{\partial \vec{b}}{\partial x}\right)$$

$$= (\vec{a}\cdot\nabla)\vec{b} + \vec{a}\times\text{curl}\,\vec{b} \quad \text{— (2)}$$

Interchanging $\vec{a}$ and $\vec{b}$ we get

$$\Sigma\left(\vec{b}\cdot\frac{\partial \vec{a}}{\partial x}\right)\hat{i} = (\vec{b}\cdot\nabla)\vec{a} + \vec{b}\times\text{curl}\,\vec{a} \quad \text{— (3)}$$

From (2) & (3), Putting in (1), we get

$$\text{grad}\,(\vec{a}\cdot\vec{b}) = \vec{a}\times\text{curl}\,\vec{b} + \vec{b}\times\text{curl}\,\vec{a} + (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a}$$

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Q A vector field is given by $\vec{A} = (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j}$. Show that the field is irrotational and find the scalar potential.

Sol Field A is irrotational if Curl $\vec{A} = \vec{0}$

Now Curl $\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$

$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy-2xy) = \vec{0}$

Field $\vec{A}$ is irrotational

If ϕ is the Scalar ~~pothenti~~ potential then

$$\vec{A} = \text{grad}\,\phi$$

$$\Rightarrow \vec{A}\cdot d\vec{r} = (\text{grad}\,\phi)\cdot d\vec{r}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$d\phi = \vec{A}\cdot d\vec{r}$$

$$= (x^2+xy^2)dx + (y^2+x^2y)dy$$

$$= x^2dx + y^2dy + (xy^2dx + x^2y\,dy)$$

$$= x^2dx + y^2dy + d\left(\frac{x^2y^2}{2}\right)$$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^3}{2} + C$$

Example: A fluid motion is given by

$$\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

Is this motion irrotational? If so find the velocity potential.

Soln We have $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

(i) The motion is irrotational if $\text{Curl}\,\vec{V} = \vec{0}$

$$\text{Curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = \vec{0}$$

Hence motion is irrotational.

Now let $\vec{V} = \text{grad}\,\phi$ where ϕ is velocity potential

$$\Rightarrow \vec{V}\cdot d\vec{r} = \nabla\phi \cdot d\vec{r}$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$d\phi = \vec{V}\cdot d\vec{r}$$

$$= \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right],$$

$$= (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$$

$$= (y\,dx + x\,dy) + (z\,dx + x\,dz) + (z\,dy + y\,dz)$$

$$= d(yx) + d(zx) + d(zy)$$

Integrating $\Rightarrow \phi = xy + zx + yz + C$

Ex If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that

(I) $\operatorname{div}(\vec{r}\,\phi) = 3\phi + \vec{r}\cdot\operatorname{grad}\phi$ (II) $\operatorname{div}(\hat{r}) = \frac{2}{r}$

Soln We know that $\operatorname{div}(u\vec{a}) = u\operatorname{div}\vec{a} + \vec{a}\cdot\operatorname{grad}u$.

(I) $\operatorname{div}(\phi\vec{r}) = \phi\operatorname{div}\vec{r} + \vec{r}\cdot\operatorname{grad}\phi$

$$= 3\phi + \vec{r}\cdot\operatorname{grad}\phi \qquad [\operatorname{div}\vec{r} = 3]$$

(II) $\operatorname{div}(\hat{r}) = \operatorname{div}\left(\frac{\vec{r}}{r}\right)$

$$= \frac{1}{r}\operatorname{div}\vec{r} + \vec{r}\cdot\operatorname{grad}\frac{1}{r}$$

$$= \frac{3}{r} + \vec{r}\left(-\frac{\hat{r}}{r^2}\right)$$

$$= \frac{3}{r} + \vec{r}\left(\frac{-\vec{r}}{r^3}\right) = \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

(34)

Q.* Prove that $\text{div}(\text{grad } r^n) = \nabla^2(r^n) = n(n+1) r^{n-2}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$.

Hence or otherwise evaluate $\nabla \times \left(\frac{\vec{r}}{r^2}\right)$

Sol^n Here $r^2 = x^2 + y^2 + z^2 \Rightarrow 2\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$$2\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad } r^n = n r^{n-1}\hat{r} = n r^{n-1}\frac{\vec{r}}{r} = n r^{n-2}\vec{r}$$

$$\text{div}(\text{grad } r^n) = \text{div}(n r^{n-2}\vec{r})$$

$$= n r^{n-2}\,\text{div}\,\vec{r} + \text{grad}(n r^{n-2})\cdot\vec{r}$$

$$= 3n r^{n-2} + n(n-2) r^{n-3}\,\hat{r}\cdot\vec{r}$$

$$= 3n r^{n-2} + n(n-2) r^{n-4}(\vec{r}\cdot\vec{r})$$

$$= 3n r^{n-2} + n(n-2) r^{n-2}$$

$$\Rightarrow \nabla^2(r^n) = n(n+1) r^{n-2}$$

Putting $n = -1$, we get

$$\nabla^2\left(\frac{1}{r}\right) = 0$$

We know that

$$\text{Curl}(u\vec{a}) = u\,\text{Curl}\,\vec{a} + (\text{grad } u)\times\vec{a}$$

$$\text{Curl}\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}\text{Curl}\,\vec{r} + \left(\text{grad}\frac{1}{r^2}\right)\times\vec{r} \qquad (\text{Curl } \vec{r} = 0)$$

$$= \vec{0} - \left(\frac{2}{r^3}\hat{r}\right)\times\vec{r} = -\frac{2}{r^4}(\vec{r}\times\vec{r}) = \vec{0}$$

Q 46 Prove that vector $f(r)\vec{r}$ is irrotational

Sol^n $\text{Curl}\{f(r)\vec{r}\} = f(r)\,\text{Curl}\,\vec{r} + \{\text{grad } f(r)\}\times\vec{r}$

$= \vec{0} + f'(r)\,\hat{r}\times\vec{r}$ $\quad (\text{Curl}\,\vec{r} = \vec{0})$

$= \frac{f'(r)}{r}(\vec{r}\times\vec{r}) = \vec{0}$ $\quad (\vec{r}\times\vec{r} = \vec{0})$

(II) Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$

Hence evaluate $\nabla^2(\log r)$ if $r = (x^2+y^2+z^2)^{1/2}$

Sol^n $\text{grad } f(r) = f'(r)\,\hat{r} = \frac{1}{r}f'(r)\vec{r}$

$\text{div}\,[\text{grad } f(r)] = \text{div}\left[\frac{f'(r)}{r}\vec{r}\right]$

$= \frac{f'(r)}{r}\,\text{div}\,\vec{r} + \text{grad}\left[\frac{f'(r)}{r}\right]\cdot\vec{r}$

$= \frac{3}{r}f'(r) + \left[\frac{r f''(r) - f'(r)}{r^2}\right]\hat{r}\cdot\vec{r}$

$= \frac{3}{r}f'(r) + \left[\frac{r f''(r) - f'(r)}{r^3}\right](\vec{r}\cdot\vec{r})$

$= \frac{3}{r}f'(r) + \left[\frac{r f''(r) - f'(r)}{r}\right]$ $\quad (\vec{r}\cdot\vec{r} = r^2)$

$\Rightarrow \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$

Now $\nabla^2(\log r) = -\frac{1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right) = \frac{1}{r^2} = \frac{1}{x^2+y^2+z^2}$.

* Show that the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. Find the scalar potential.

Sol^n For the vector field $\vec{F}$ to be irrotational, $\text{curl}\,\vec{F} = \vec{0}$

We know that $\text{curl}(u\vec{a}) = u\,\text{curl}\,\vec{a} + (\text{grad}\,u)\times\vec{a}$

$$\text{curl}\left(\frac{1}{r^3}\vec{r}\right) = \frac{1}{r^3}\text{curl}\,\vec{r} + \left(\text{grad}\,\frac{1}{r^3}\right)\times\vec{r} \qquad [\text{curl}\,\vec{r} = \vec{0}]$$

$$= \frac{1}{r^3}(\vec{0}) + \left(\frac{-3}{r^4}\hat{r}\right)\times\vec{r}$$

$$= \vec{0} - \frac{3}{r^5}(\vec{r}\times\vec{r}) = \vec{0} - \vec{0} = \vec{0}$$

Hence vector field is irrotational.

Again, for vector field $\vec{F}$ to be solenoidal $\text{div}\,\vec{F} = 0$

We know that $\text{div}(u\vec{a}) = u\,\text{div}\,\vec{a} + \vec{a}\cdot\text{grad}\,u$.

$$\text{div}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\text{div}\,\vec{r} + \vec{r}\cdot\text{grad}\left(\frac{1}{r^3}\right)$$

$$= \frac{3}{r^3} + \vec{r}\left(\frac{-3}{r^4}\cdot\frac{\vec{r}}{r}\right) \qquad [\text{div}\,\vec{r} = 3]$$

$$= \frac{3}{r^3} - \frac{3}{r^5}r^2 = \frac{3}{r^3} - \frac{3}{r^3} = 0$$

Hence vector field $\vec{F}$ is solenoidal.

Now let $\vec{F} = \nabla\phi$ where ϕ is scalar ~~function~~ potential

$$\vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = d\phi$$

$$d\phi = \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}} \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$\phi = \int \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$x^2+y^2+z^2 = t$

~~2x dx + dy + dz =~~

$2x\,dx + 2y\,dy + 2z\,dz = dt$

$x\,dx + y\,dy + z\,dz = \frac{1}{2}dt$

$$= \frac{1}{2}\int \frac{1}{t^{3/2}}\,dt$$

$$\phi = \frac{1}{2}\left[\frac{t^{-3/2+1}}{-3/2+1}\right]$$

$$\phi = \frac{-1}{\sqrt{t}} + C$$

$$\phi = \frac{-1}{\sqrt{x^2+y^2+z^2}} + C$$

$$\phi = \frac{-1}{\sqrt{r^2}} + C \qquad \because r = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\phi = \frac{-1}{r} + C$$

Practice Question

* Find the constans a, b, c so that $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

If $\vec{F} = \text{grad}\,\phi$

Show that $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$.

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* Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find the velocity potential ϕ such that $\vec{A} = \nabla\phi$

* A fluid motion is given by $\vec{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$. Is the motion irrotational? If so, find the velocity potential.

Line Integral Question

* A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the line integral over the circular path given by $x^2+y^2 = a^2$, $z=0$

Sol Put $x = a\cos\theta$, $y = a\sin\theta$, $z=0$, $\theta \to 0$ to 2π

Since the particle moves in xy-plane, $z=0$

$$\vec{r} = x\hat{i} + y\hat{j} \Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\oint \vec{F}\cdot d\vec{r} = \oint [\sin y\,\hat{i} + x(1+\cos y)\hat{j}][dx\hat{i} + dy\hat{j}]$$

$$= \oint_C \sin y\,dx + x(1+\cos y)\,dy$$

$$= \oint [(\sin y\,dx + x\cos y\,dy) + x\,dy]$$

$$= \oint_C d(x\sin y) + \oint x\,dy$$

$$= \int_0^{2\pi} d[a\cos\theta\sin(a\sin\theta)] + \int_0^{2\pi} a\cos\theta\, a\cos\theta\,d\theta$$

$$= [a\cos\theta\sin(a\sin\theta)]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}(2\pi) = \pi a^2$$

Line integrals

Any integral which is to be evaluated along a curve is called line integral. **Note** Total work done by $\vec{F}$ during displacement from A to B is $\int_A^B \vec{F}\cdot d\vec{R}$

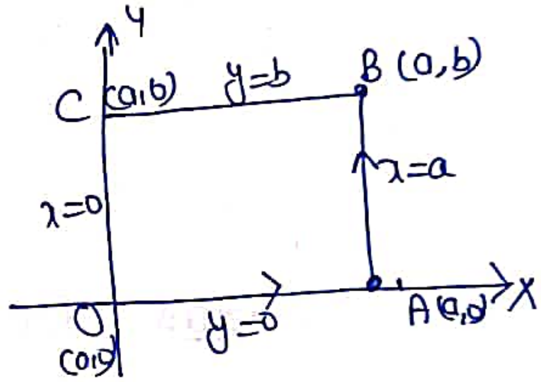
* Find the total work done by a force $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ in moving a point $(0,0)$ to (a,b) along the rectangle bounded by lines $x=0$, $x=a$, $y=0$ and $y=b$

Soln

$$\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy$$



Total work done

$$= \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r}$$

$$= I_1 + I_2$$

where $I_1 = \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$ [Along OA, $y=0$, $dy=0$]

$I_2 = \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b -2ay\,dy = -ab^2$ [Along AB, $x=a$: $dx=0$]

Total work done $= \frac{a^3}{3} - ab^2$.

Eg If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$

$$\vec{F}\cdot d\vec{r} = 3xy\,dx - y^2\,dy$$
$$= 3x(2x^2)\,dx - 4x^4\,(2(2x)dx)$$
$$= 6x^3\,dx - 16x^5\,dx$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (6x^3\,dx - 16x^5)\,dy$$
$$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

Green's Theorem in the Plane

If C is a regular closed curve in the xy-plane and R be the region bounded by C, then

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Where $M(x,y)$ and $N(x,y)$ are continuously differentiable function inside and on C.

* By the use of Green's theorem, show that area bounded by a simple closed curve C is given by $\frac{1}{2}\int(x\,dy - y\,dx)$ Hence find the area of an ellipse.

Sol^n By Green theorem in planes, we have.

$$\int(M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—(1)}$$

Put $M = -y$, $N = x$.

$\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$

Hence from (1)

$$\int_C(-y\,dx + x\,dy) = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy = 2A$$

Where A is the required area

$$A = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

Put $x = a\cos\phi$, $y = b\sin\phi$

$$A = \frac{1}{2}\int_0^{2\pi} (a\cos\phi)(b\cos\phi\,d\phi) - (b\sin\phi)(-a\sin\phi)\,d\phi$$

$$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2\phi + \sin^2\phi)\,d\phi = \frac{1}{2}ab(2\pi) = \pi ab.$$

* If C is a simple closed curve in the xy-plane not containing the origin, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \dfrac{-\hat{i}y + \hat{j}x}{x^2+y^2}$

Sol. Let $\vec{F} = M\hat{i} + N\hat{j}$ then

$$\vec{F}\cdot d\vec{r} = (M\hat{i} + N\hat{j})(dx\,\hat{i} + dy\,\hat{j}) = M\,dx + N\,dy$$

For given $\vec{F}$, we have

$$M = \frac{-y}{x^2+y^2}, \quad N = \frac{x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Hence from Green's theorem, we have.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

$$= 0$$

Ex A vector field $\vec{F}$ is given by $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the integral $\int_C \vec{F}\cdot d\vec{a}$ where C is the circular path given by $x^2+y^2=a^2$

Sol^n $\int_C \vec{F}\cdot d\vec{a} = \int_C (\sin y\,\hat{i} + x(1+\cos y)\hat{j})(dx\,\hat{i} + dy\,\hat{j})$

$$= \int [\sin y\,dx + x(1+\cos y)\,dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x} x(1+\cos y) - \frac{\partial}{\partial y}\sin y\right] dx\,dy \quad \text{by Green's thm.}$$

$$= \iint_R (1+\cos y - \cos y)\,dx\,dy = \iint_R dx\,dy$$

$$= A = \text{area of circle}$$

$$= \pi(\text{radius})^2 = \pi a^2.$$

* Apply Green's theorem to evaluate $\int_C [(2x^2-y^2)dx + (x^2+y^2)dy]$ where C is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$

Sol^n Let $\vec{F} = M\hat{i} + N\hat{j}$ then

$$\vec{F}\cdot d\vec{a} = (M\hat{i} + N\hat{j})(dx\,\hat{i} + dy\,\hat{j}) = M\,dx + N\,dy$$

For given $\vec{F}$, we have

$$M = 2x^2 - y^2, \quad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = 2x.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$$

$$= \iint_R (2x+2y)\,dx\,dy$$

$$= 2\int_{x=-a}^{a}\int_0^{\sqrt{a^2-x^2}} (x+y)\,dx\,dy$$

$$= 2\int_{-a}^{a}\left(xy+\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx.$$

$$= 2\int_{-a}^{a}\left(x\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}\right)dx$$

$$= 2\int_0^a (a^2-x^2)\,dx = 2\left(a^2x-\frac{x^3}{3}\right)_0^a$$

$$= 2\left(a^3-\frac{a^3}{3}\right) = \frac{4}{3}a^3$$

imp **Example** Use Green's theorem to evaluate $\int_C (x^2+xy)dx + (x^2+y^2)dy$ where C is the square formed by line $y=\pm 1$, $x=\pm 1$

Sol^n $\int_C (x^2+xy)dx + (x^2+y^2)dy = \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(x^2+xy)\right] dx\,dy$

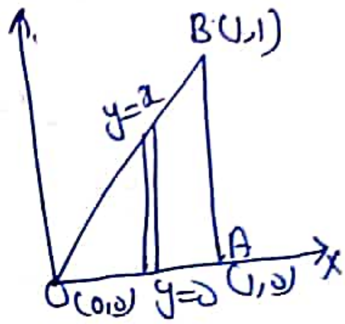
$$= \iint_R (2x-x)\,dx\,dy.$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1} x\,dx\,dy = \int_{-1}^{1} x(y)_{-1}^{1} dx$$

$$= \int_{-1}^{1} 2x\,dx = 0$$

* Using Green's theorem $\int_C (x^2y\,dx + x^2dy)$, where C is the boundary described counter clockwise of the triangle with vertices (0,0), (1,0), (1,1).

Sol^n $\int_C (x^2y\,dx + x^2dy) = \iint_R \left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(x^2y)\right] dx\,dy$



$$= \iint_R \left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(x^2y)\right] dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} (2x - x^2)\,dx\,dy$$

$$= \int_0^1 (2x^2 - x^3)\,dx = \left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$

Ex Evaluate by Green's theorem $\int_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi, \pi/2)$, $(0, \pi/2)$ and hence verify Green's theorem.

Sol^n Given integral $= \iint_R \left[\frac{\partial}{\partial x}(e^{-x}\cos y) - \frac{\partial}{\partial y}(e^{-x}\sin y)\right] dx\,dy$

$$= \int_{x=0}^{\pi}\int_{y=0}^{\pi/2} (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy.$$

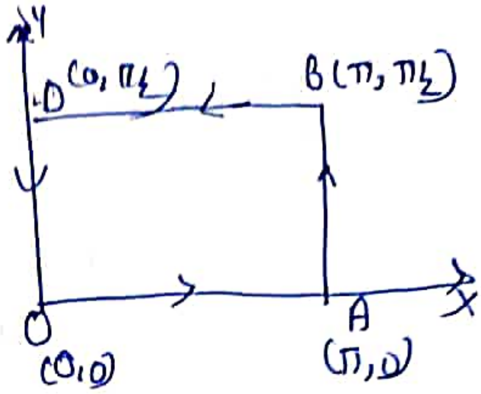
$$= -2\int_0^{\pi} e^{-x}\,dx \int_0^{\pi/2} \cos y\,dy$$

$$= -2(-e^{-x})_0^{\pi}(\sin y)_0^{\pi/2} = 2(e^{-\pi} - 1)$$

Verification of Theorem

For this purpose, let us evaluate the given line integral directly.

$$\int_C \left(e^{-x}\sin y\, dx + e^{-x}\cos y\, dy\right)$$



$$= \int_{OA} \left(e^{-x}\sin y\, dx + e^{-x}\cos y\, dy\right)$$

$$+ \int_{AB} \left(e^{-x}\sin y\, dx + e^{-x}\cos y\, dy\right)$$

$$+ \int_{BD} \left(e^{-x}\sin y\, dx + e^{-x}\cos y\, dy\right) + \int_{DO} \left(e^{-x}\sin y\, dx + e^{-x}\cos y\, dy\right)$$

Now, along OA, $y = 0 \Rightarrow dy = 0$

along AB, $x = \pi \Rightarrow dx = 0$

BD, $y = \pi/2 \Rightarrow dy = 0$

DO, $x = 0 \Rightarrow dx = 0$

Hence the given line integral

$$= 0 + \int_0^{\pi/2} e^{-\pi}\cos y\, dy + \int_{\pi}^{0} e^{-x}\, dx + \int_{\pi/2}^{0} \cos y\, dy$$

$$= e^{-\pi}\left(\sin y\right)_0^{\pi/2} + \left(-e^{-x}\right)_{\pi}^{0} + \left(\sin y\right)_{\pi/2}^{0}$$

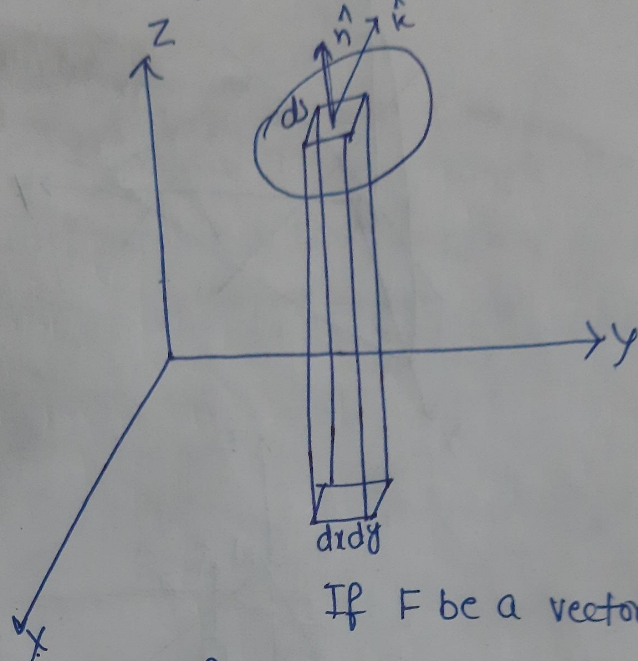
$$= e^{-\pi} - (1 - e^{-\pi}) + (-1)$$

$$= 2(e^{-\pi} - 1)$$

Hence Green's theorem verified.

Surface Integrals

Any integral which is to be evaluated over a surface is called surface integral.



If F be a vector point function then surface integral is denoted by $\iint_S \vec{F}\cdot\hat{n}\,ds$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}\cdot d\vec{s}$$

Now dxdy = Projection of ds on the xy-plane.

then
$$ds = \frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$$

$$\iint_R \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$$

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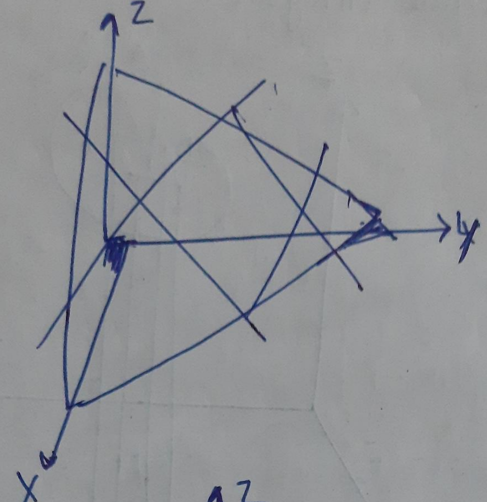
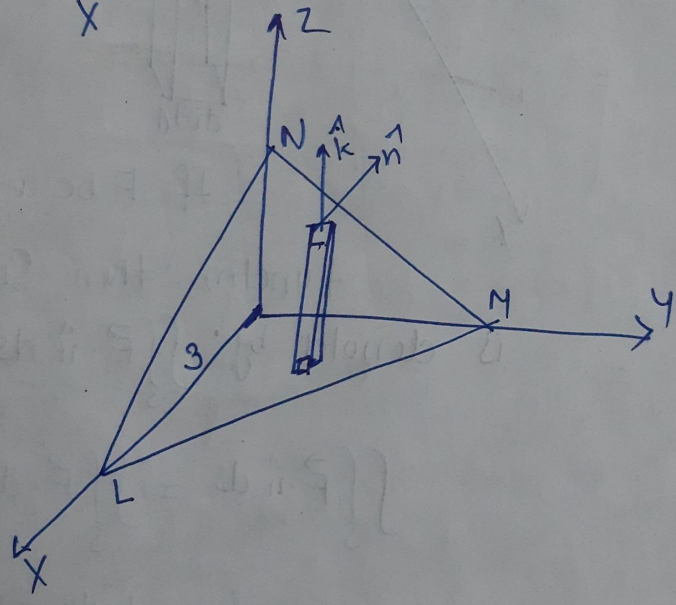
* Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ Where $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the Surface of the plane $2x+y+2z=6$ in the first octant.

Sol^n





Sol^n A vector normal to the Surface S is given by $\nabla(2x+y+2z) = 2\hat{i}+\hat{j}+2\hat{k}$

$\hat{n}$ = a Unit vector normal to Surface

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{(2)^2+(1)^2+(2)^2}} = \frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$$\hat{k}\cdot\hat{n} = \hat{k}\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$$

$$\iint \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$$

Where R is the Projection of S ie triangle LMN on the xy-plane. The region R ie triangle OLM is bounded by x-axis, y axis and line $2x+y=6$, $z=0$

$$\text{Now } \vec{A}\cdot\hat{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$= \frac{2}{3}y\,(y + 6 - 2x - y)$$

$$= \frac{4}{3}y\,(3-x)$$

$$\text{Hence } \iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$$

$$= \int_0^3\int_0^{6-2x} \frac{8}{9}y(3-x)\,dy\,dx.$$

$$= \int_0^3 \frac{8}{9}(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx.$$

$$= \frac{4}{9}\int_0^3 (3-x)(6-2x)^2\,dx.$$

$$= \frac{16}{9}\int_0^3 (3-x)^3\,dx = \frac{16}{9}\left[\frac{(3-x)^4}{4(-1)}\right]_0^3 = \frac{4}{9}(81) = 36$$

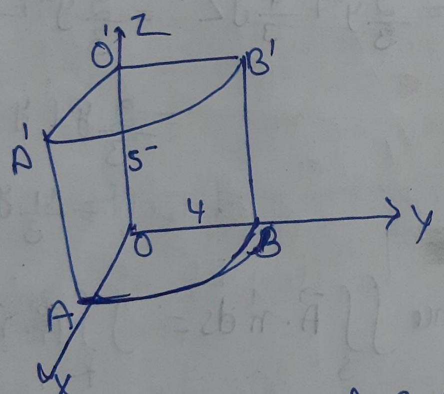
S0

Ex Evaluate $\iint_S \vec{A}\cdot\hat{n}\, ds$ where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Soln A vector normal to the surface S is given by $\nabla(x^2+y^2-16) = 2x\hat{i} + 2y\hat{j}$

$\hat{n}$ = a unit vector normal to surface S

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{4}$$



Let R be the projection of S on yz-plane, then

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

The region R is OBB'O' enclosed by $y=0$ to $y=4$ and $z=0$ to $z=5$

Now $\hat{i}\cdot\hat{n} = \hat{i}\cdot\left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}\right) = \frac{1}{4}x$

$\vec{A}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k})\left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}\right) = \frac{1}{4}x(y+z)$

Hence $\iint_R \vec{A}\cdot\hat{n}\, ds = \iint_R \vec{A}\cdot\hat{n} \frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$

$$= \iint_R \frac{1}{4}x(y+z) \frac{dy\,dz}{\frac{1}{4}x}.$$

$$= \int_0^5 \int_0^4 (y+z)\, dy\, dz.$$

$$= \int_0^5 \left[\frac{y^2}{2} + zy\right]_0^4 dz = \int_0^5 (8 + 4z)\, dz$$

$$= \left[8z + \frac{4z^2}{2}\right]_0^5$$

$$= 40 + 50 = 90$$

Ex If $\vec{F} = (2x^2 - 3z)\hat{i} - 2yx\hat{j} - 4x\hat{k}$, then evaluate $\iiint_V \nabla\cdot\vec{F}\, dV$, where V is bounded by ~~the~~ the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$

Solⁿ $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 4x - 2x = 2x.$

$$\iiint_V \nabla\cdot\vec{F}\, dV = \iiint_V 2x\, dx\, dy\, dz$$

$$= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2x\, dz\, dy\, dx.$$

$$= \int_0^2 \int_0^{2-x} 2x\,[z]_0^{4-2x-2y}\, dy\, dx.$$

$$= \int_0^2 \int_0^{2-x} 2x(4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \left[4x(2-x)y - 2xy^2\right]_0^{2-x} dx.$$

$$= \int_0^2 2x(2-x)^2\,dx = 2\int_0^2 (4x-4x^2+x^3)\,dx$$

$$= 2\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2$$

$$= 2\left(8 - \frac{32}{3} + 4\right) = \frac{8}{3}$$

Gauss-Divergence Theorem (Relation between Surface integral and volume integral)

If $\vec{F}$ is a vector point function having continuous first order partial derivatives in the region V bounded by a closed surface S, then.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dv.$$

where $\hat{n}$ is the outward drawn unit normal vector to the surface S.

Example For any closed surface S, prove that $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 0$

Soln By divergence theorem, we have

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iiint_V (\text{div curl}\,\vec{F})\,dv = 0$$

Ex Evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$, where S is a closed surface

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Soln By Gauss' divergence theorem

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{r}\,dV$$

$$= \iiint_V 3\,dv = 3V$$

Where V is the volume enclosed by S.

Ex The vector field $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ is defined over the volume of the cuboid given by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ enclosing the surface, evaluate $\iint_S \vec{F}\cdot d\vec{S}$

Sol^n By Gauss-divergence theorem.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\, dV.$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (2x+y)\, dV$$

$$= \int_0^a \int_0^b \int_0^c (2x+y)\, dz\, dy\, dx.$$

$$= \int_0^a \int_0^b (2x+y)(z)_0^c\, dy\, dx.$$

$$= c \int_0^a \left(2xy + \frac{y^2}{2}\right)_0^b dx.$$

$$= bc \int_0^a \left(2x + \frac{b}{2}\right) dx = bc\left(x^2 + \frac{b}{2}x\right)_0^a$$

$$= abc\left(a + \frac{b}{2}\right)$$

* Evaluate $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).\hat{n}\, ds$, where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane and bounded by this plane.

Sol

Sol^n Let V be the volume enclosed by the surface S.

Then by divergence theorem, we have.

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}).\hat{n}\, ds = \iiint_V \operatorname{div}(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dv$$

$$= \iiint_V 2zy^2\, dv = 2\iiint_V zy^2\, dv$$

$x \Rightarrow r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$dv = r^2\sin\theta\, dr\, d\theta\, d\phi$

To cover V, the limit of r will be 0 to 1, $\theta \to 0$ to $\frac{\pi}{2}$, $\phi \to 0$ to 2π

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r^2\sin^2\theta\sin^2\phi) r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5\sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta\cos\theta\sin^2\phi\left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi}\sin^2\phi\,\frac{2}{4.2}\,d\phi = \frac{1}{12}\int_0^{2\pi}\sin^2\phi\, d\phi$$

$$= \frac{\pi}{12}.$$

* Apply divergence theorem to evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 4x^3\hat{i} - x^2y\hat{j} + x^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=a^2$ bounded by the planes $z=0$ and $z=b$

Soln $\vec{F} = 4x^3\hat{i} - x^2y\hat{j} + x^2z\hat{k}$

$$\operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4x^3\hat{i} - x^2y\hat{j} + x^2z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4x^3) + \frac{\partial}{\partial y}(-x^2y) + \frac{\partial}{\partial z}(x^2z)$$

$$= 12x^2 - x^2 + x^2 = 12x^2.$$

$$\iiint_V \operatorname{div}\vec{F}\, dv = 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2\, dz\, dy\, dx.$$

$$= 12\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 (z)_0^b\, dy\, dx.$$

$$= 12b\int_{-a}^{a} x^2 (y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\, dx.$$

$$= 12b\int_{-a}^{a} x^2 \cdot 2\sqrt{a^2-x^2}\, dx.$$

$$= 24b\int_{-a}^{a} x^2\sqrt{a^2-x^2}\, dx.$$

Put $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$.

$$= 48b\int_{0}^{a} x^2\sqrt{a^2-x^2}\, dx.$$

$$= 48b\int_{0}^{\pi/2} a^2\sin^2\theta \cdot a\cos\theta \cdot a\cos\theta\, d\theta.$$

$$48ba^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta = 48ba^4\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\Gamma 3}$$

$$= 48ba^4\,\frac{\frac{\sqrt{\pi}}{2}\,\frac{\sqrt{\pi}}{2}}{2\cdot 2}$$

$$= 3ba^4\pi.$$

(**) Evaluate $\iint_S (a^2x^2+b^2y^2+c^2z^2)^{1/2}\,ds$ where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$

Sol

Let $\phi = ax^2+by^2+cz^2-1$ —— (1)

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(ax^2+by^2+cz^2-1)$$

$$= 2ax\hat{i}+2by\hat{j}+2cz\hat{k} = 2(ax\hat{i}+by\hat{j}+cz\hat{k})$$

$$|\text{grad}\,\phi| = 2\sqrt{a^2x^2+b^2y^2+c^2z^2}.$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2(ax\hat{i}+by\hat{j}+cz\hat{k})}{2\sqrt{a^2x^2+b^2y^2+c^2z^2}} \quad —— (2)$$

According to question

$$\vec{F}\cdot\hat{n} = \sqrt{a^2x^2+b^2y^2+c^2z^2}. \quad —— (3)$$

Consequently $\vec{F} = ax\hat{i}+by\hat{j}+cz\hat{k}$

$$\text{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(ax\hat{i}+by\hat{j}+cz\hat{k})$$

$$= a+b+c.$$

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Now, $\iiint_V \text{div}\,\vec{F}\,dV = (a+b+c)\iiint_V dV$ — (4)

Volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4}{3}\pi abc$.

Volume of ellipsoid $(ax^2+by^2+cz^2=1)$ is $\frac{4}{3}\pi\frac{1}{\sqrt{a}}\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}$

$$= \frac{4\pi}{3\sqrt{abc}}.$$

from Eq^n (4) $\iiint_V \text{div}\,\vec{F}\,dv = (a+b+c)\frac{4\pi}{3\sqrt{abc}}.$

$$= \frac{4\pi(a+b+c)}{3\sqrt{abc}}$$

Example Evaluate $\iint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\,d\bar{s}$ where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$

Sol^n Let $\vec{F}\cdot\hat{n} = (a^2x^2+b^2y^2+c^2z^2)^{-1/2}.$ — (1)

Let $\phi = ax^2+by^2+cz^2-1$

$\text{grad}\,\phi = 2ax\,\hat{i}+2by\,\hat{j}+2cz\,\hat{k}.$

$|\text{grad}\,\phi| = 2\sqrt{a^2x^2+b^2y^2+c^2z^2}$

$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{ax\hat{i}+by\hat{j}+cz\hat{k}}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}.$ — (2)

from (1) and (2) it is clear that

$\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$

$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$

$= 1+1+1 = 3$

By $\iint_S \vec{F}\cdot d\bar{s} = \iiint_V \text{div}\,\vec{F}\,dv = 3\iiint_V dV = 3V \Rightarrow \left(\frac{4\pi}{3}\frac{1}{\sqrt{abc}}\right)$

* Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

Soln $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ — (1)

$$\operatorname{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$$

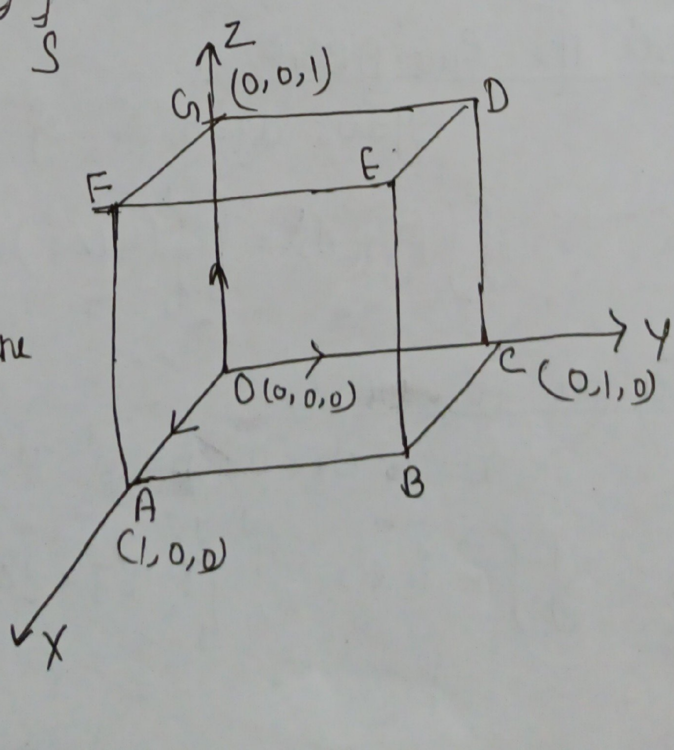
$$\iiint_V \operatorname{div}\vec{F}\, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z - y)\, dz\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left(2z^2 - yz\right)_0^1 dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\, dy\, dx$$

$$= \int_{x=0}^{1}\left(2y - \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}$$ — (2)

To evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$

Here S is the surface of cube bounded by the 6 plane surfaces



Over the face OABC

$z=0,\ dz=0,\ \hat{n}=-\hat{k}\quad ds=dx\,dy$

$$\int \vec{F}\cdot\hat{n}\,ds=\int_0^1\int_0^1(-y^2\hat{j})\cdot(-\hat{k})\,dx\,dy=0 \quad —(3)$$

Over the face BCDE

$y=1\quad dy=0,\ \hat{n}=\hat{j}\quad ds=dx\,dz$

$$\iint_D \vec{F}\cdot\hat{n}\,ds=\int_0^1\int_0^1(4xz\hat{i}-\hat{j}+z\hat{k})\cdot\hat{j}\,dx\,dz$$

$$=-\int_0^1\int_0^1 dx\,dz=-(x)_0^1(z)_0^1=-1 \quad —(4)$$

Over the face DEFG

$z=1,\ dz=0,\ \hat{n}=\hat{k},\ ds=dx\,dy$

$$\iint \vec{F}\cdot\hat{n}\,ds=\int_0^1\int_0^1(4x\hat{i}-y^2\hat{j}+y\hat{k})\cdot\hat{k}\,dx\,dy$$

$$\int_0^1\int_0^1 y\,dx\,dy=\int_0^1 dx\int_0^1 y\,dy=(x)_0^1\left(\frac{y^2}{2}\right)_0^1=\frac{1}{2} \quad —(5)$$

Over the face AOGF

$y=0,\ dy=0,\ \hat{n}=-\hat{j}\quad ds=dx\,dz$

$$\iint \vec{F}\cdot\hat{n}\,ds=\int_0^1\int_0^1(4xz\hat{i})\cdot(-\hat{j})\,dx\,dz=0 \quad —(6)$$

Over the face OCDG

$x=0,\ dx=0,\ \hat{n}=-\hat{i}\quad ds=dy\,dz$

$$\iint \vec{F}\cdot\hat{n}\,ds=\int_0^1\int_0^1(-y^2\hat{j}+yz\hat{k})(-\hat{i})\,dy\,dz=0 \quad —(7)$$

Over the face ABEF

$$x=1,\ dx=0,\ \hat{n}=\hat{i},\ ds=dy\,dz$$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 \left(4z\hat{i}-y^2\hat{j}+yz\hat{k}\right)\cdot\hat{i}\,dy\,dz$$

$$=\int_0^1\int_0^1 4z\,dy\,dz = \int_0^1 dy\int_0^1 4z\,dz = (y)_0^1\,(2z^2)_0^1 = 2 \quad \text{—} (8)$$

Adding (3), (4), (5), (6), (7), (8), we get over the whole Surface $\iint \vec{F}\cdot\hat{n}\,ds = 0-1+\frac{1}{2}+0+0+2 = \frac{3}{2}$ — (9)

from eqn (2) and (9) $\iint \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$

Hence divergence theorem verified.

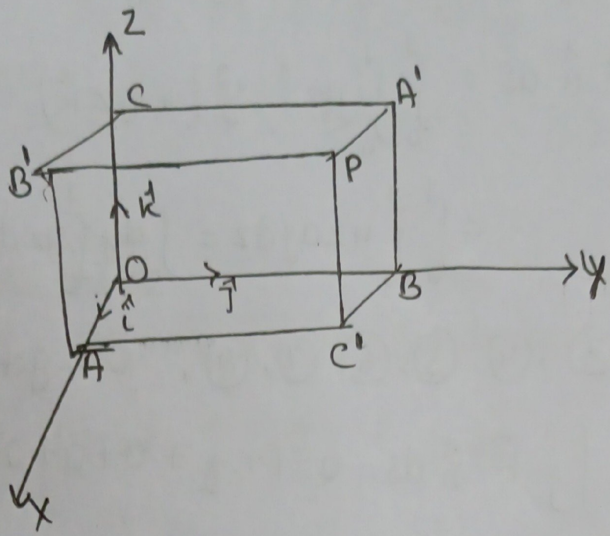
Ex Verify divergence theorem for $\vec{F}=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ taken over the rectangular parallelopiped $0\le x\le a,\ 0\le y\le b,\ 0\le z\le c$.

Soln For the verification of divergence theorem, we shall evaluate volume and surface integral separately and show that they are equal.

Now $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2-yz)+\frac{\partial}{\partial y}(y^2-zx)+\frac{\partial}{\partial z}(z^2-xy)$

$$=2x+2y+2z$$

$$=2(x+y+z)$$



$$\iiint_V \text{div}\, \vec{F}\, dV = \int_0^c \int_0^b \int_0^a 2(x+y+z)\, dx\, dy\, dz$$

$$= \int_0^c \int_0^b 2\left[\frac{x^2}{2} + yx + zx\right]_0^a dy\, dz$$

$$= \int_0^c \int_0^b 2\left(\frac{a^2}{2} + ya + za\right) dy\, dz$$

$$= \int_0^c 2\left[\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right) dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c) \quad \text{--- } (1)$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelopiped into 6 parts.

$S_1 =$ the face $OAC'B$

$S_2 =$ the face $CB'PA'$

$S_3 =$ the face $OBA'C$

$S_4 =$ the face $AC'PB'$

$S_5 =$ the face $OCB'A$

$S_6 =$ the face $BA'PC'$

Also
$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds + \iint_{S_3} \vec{F}\cdot\hat{n}\,ds + \iint_{S_4} \vec{F}\cdot\hat{n}\,ds$$
$$+ \iint_{S_4} \vec{F}\cdot\hat{n}\,ds + \iint_{S_4} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds$$

On S_1 $(z=0)$, we have $\hat{n} = -\hat{k}$

$$\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$$

So that $\vec{F}\cdot\hat{n} = (x^2\hat{i} + y^2\hat{j} - xy\hat{k})(-\hat{k}) = xy$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a xy\,dx\,dy = \int_0^b\left[y\frac{x^2}{2}\right]_0^a dy = \frac{a^2}{2}\int_0^b y\,dy = \frac{a^2b^2}{4}$$

On S_2 $(z=c)$, we have $\hat{n} = \hat{k}$

$$\vec{F} = (x^2-cy)\hat{i} + (y^2-cx)\hat{j} + (c^2-xy)\hat{k}$$

So that $\vec{F}\cdot\hat{n} = [(x^2-cy)\hat{i} + (y^2-cx)\hat{j} + (c^2-xy)\hat{k}]\cdot\hat{k}$

$$= c^2 - xy.$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (c^2-xy)\,dx\,dy = \int_0^b \left(c^2a - \frac{a^2y}{2}\right)dy = abc^2 - \frac{a^2b^2}{4}$$

On S_3 $(x=0)$ we have $\hat{n}=-\hat{i}$

$$\vec{F} = -yz\hat{i} + y^2\hat{j} + z^2\hat{k}$$

$$\vec{F}\cdot\hat{n} = \left(-yz\hat{i} + y^2\hat{j} + z^2\hat{k}\right)(-\hat{i}) = yz$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b yz\,dy\,dz = \int_0^c \frac{b^2}{2}z\,dz = \frac{b^2c^2}{4}$$

On S_4 $(x=a)$, we have

$$\hat{n}=\hat{i},\ \vec{F} = (a^2-yz)\hat{i} + (y^2-az)\hat{j} + (z^2-ay)\hat{k}$$

So that $\vec{F}\cdot\hat{n} = [(a^2-yz)\hat{i} + (y^2-az)\hat{j} + (z^2-ay)\hat{k}]\cdot\hat{i}$

$$= a^2-yz$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (a^2-yz)\,dy\,dz = \int_0^c\left(a^2b - \frac{b^2}{2}z\right)dz$$

$$= a^2bc - \frac{b^2c^2}{4}$$

On S_5 $(y=0)$ we have $\hat{n}=-\hat{j}$, $\vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$

So that $\vec{F}\cdot\hat{n} = (x^2\hat{i} - zx\hat{j} + z^2\hat{k})(-\hat{j}) = zx.$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^c zx\,dz\,dx = \int_0^a \frac{c^2}{2}x\,dx = \frac{a^2c^2}{4}.$$

On S_6 $(y=b)$, we have $\hat{n}=\hat{j}$, $\vec{F} = (x^2-bz)\hat{i} + (b^2-zx)\hat{j} + (z^2-bx)\hat{k}$

So that $\vec{F}\cdot\hat{n} = [(x^2-bz)\hat{i} + (b^2-zx)\hat{j} + (z^2-bx)\hat{k}]\cdot\hat{j} = b^2-zx.$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^c (b^2-zx)\,dz\,dx = \int_0^a\left(b^2c - \frac{c^2}{2}x\right)dx = ab^2c - \frac{a^2c^2}{4}.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{a^2b^2}{4} + a^2bc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

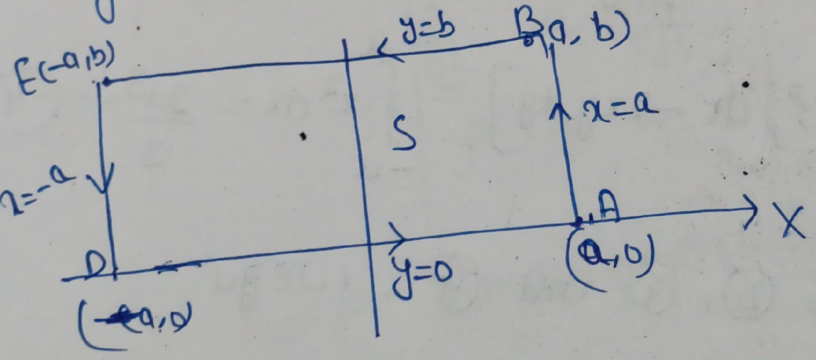
$$= abc(a+b+c) \quad \text{Hence verified.}$$

Stock's Theorem

If S is an open Surface bounded by a Closed Curve C and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is any vector point function having Continuous first Order Partial derivative then

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, ds}$$

Ex Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a,\ y = 0,\ y = b$



Sol^n Let C denote the boundary of rectangle ABCD

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C [(x^2+y^2)\hat{i} - 2xy\hat{j}]\,[\hat{i}\,dx + \hat{j}\,dy]$$

$$= \oint_C (x^2+y^2)\,dx - 2xy\,dy$$

The Curve C Consists of four lines AB, BE, ED and DA

Along AB, $x = a$, $dx = 0$, $y \to 0$ to b

$$\int_{AB} [(x^2+y^2)\,dx - 2xy\,dy] = \int_0^b -2ay\,dy = -a[y^2]_0^b = -ab^2 \quad (1)$$

Along BE, $y=b$, $dy=0$, $x \to 0$ to $-a$.

$$\int_{BE} \left[(x^2+y^2)dx - 2xy\,dy\right] = \int_{-a}^{a} (x^2+b^2)\,dx = \left[\frac{x^3}{3}+b^2x\right]_{a}^{-a}$$

$$= -\frac{2a^3}{3} - 2ab^2 \quad (2)$$

Along ED, $x=-a$, $dx=0$ and $y \to b$ to 0

$$\int_{ED} \left[(x^2+y^2)dx - 2xy\,dy\right] = \int_{b}^{0} 2ay\,dy = a\left[y^2\right]_b^0 = -ab^2 \quad (3)$$

Along DA, $y=0$, $dy=0$ and $x \to -a$ to a.

$$\int_{DA} \left[(x^2+y^2)dx - 2xy\,dy\right] = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3} \quad (4)$$

Adding (1), (2), (3) and (4), we get

$$\oint_C \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \quad (5)$$

Now $\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$

$$= (-2y - 2y)\hat{k} = -4y\hat{k}$$

For the Surface S, $\hat{n} = \hat{k}$

$$\text{Curl}\,\vec{F}\cdot\hat{n} = -4y\hat{k}\cdot\hat{k} = -4y.$$

$$\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = \int_0^b\int_{-a}^{a} -4y\,dx\,dy = \int_0^b -4y\,[x]_{-a}^{a}\,dy = -8a\int_0^b y\,dy$$

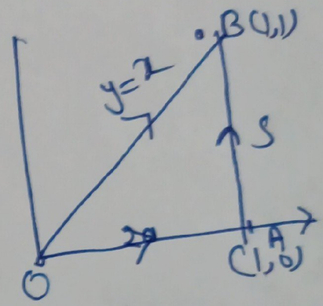
$$= -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad (6)$$

The equality (5) and (6) verifies Stoke's theorem.

Ex Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem, where ~~$\vec F = y^2\hat i + x^2\hat j$ and~~ $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$

Sol^n Since z-coordinates of each vertex of the triangle is zero, therefore triangle lies in xy-plane and $\hat{n} = \hat{k}$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$



$$= \hat{j} + 2(x-y)\hat{k}$$

$$\text{Curl}\,\vec{F}\cdot\hat{n} = [\hat{j} + 2(x-y)\hat{k}]\cdot\hat{k} = 2(x-y)$$

The equation of line OB is $y = x$.

$$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^x 2(x-y)\,dy\,dx.$$

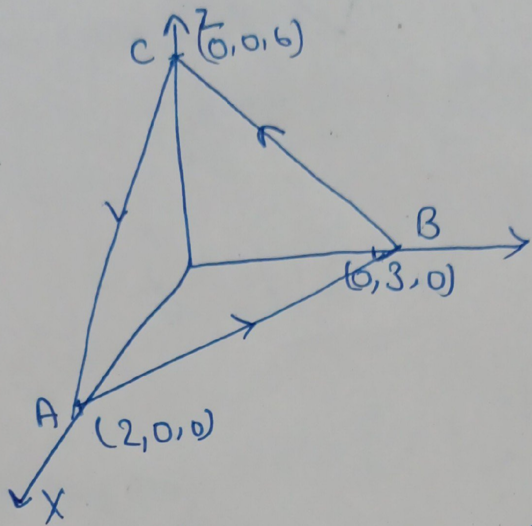
$$= \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1\left(x^2 - \frac{x^2}{2}\right)dx$$

$$= \int_0^1 x^2\,dx = \frac{1}{3}$$

* Apply Stoke's theorem to evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$

where C is the boundary of the triangle with vertices

(2, 0, 0), (0,3,0) and (0, 0, 6)



Let S be the plane surface of triangle ABC bounded by C. Let $\hat{n}$ be the unit normal vector to the surface S. Then by Stoke's theorem, we have.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds \quad \text{---} (1)$$

Here $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{n} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}(1+1) - \hat{j}(0-0) + \hat{n}(2-1)$$

$$= 2\hat{i} + \hat{k}$$

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Eqn of the plane of ~~...~~ ABC $= \frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

Let $\phi = \frac{x}{2}+\frac{y}{3}+\frac{z}{6}-1$

Normal to the plane $\triangle ABC$ is

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}-1\right)$$

$$= \frac{\hat{i}}{2}+\frac{\hat{j}}{3}+\frac{\hat{k}}{6}$$

Unit normal vector $(\hat{n}) = \dfrac{\frac{\hat{i}}{2}+\frac{\hat{j}}{3}+\frac{\hat{k}}{6}}{\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{36}}}$

$$= \frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})$$

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S (2\hat{i}+\hat{k})\cdot\frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})\frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

$$= \iint_S \frac{7}{\sqrt{14}}\frac{dx\,dy}{\left(\frac{1}{\sqrt{14}}\right)} = 7\iint_S dx\,dy$$

$= 7$ (Area of $\triangle AOB$)

$= 7\left(\frac{1}{2}\times 2\times 3\right) = 21$